問題 3.4 の解答

問題 3.4 の 1(定理 3.4.3) (2)(3) は難, 後回しでよい

(1) $\displaystyle\int_0^1 \sqrt{1+4x^2}dx = \frac{1}{2}\int_0^2 \sqrt{1+t^2}dt \quad (2x=t)$

$\displaystyle= \Big[\frac{1}{4}(t\sqrt{1+t^2} + \log|\, t+\sqrt{1+t^2}\,|)\Big]_0^2$ (p.60)

$\displaystyle= \frac{\sqrt{5}}{2} + \frac{1}{4}\log(2+\sqrt{5})$

(2) 長さ $\displaystyle= \int_1^a \sqrt{1+\frac{1}{x^2}}dx = \int_1^a \frac{\sqrt{1+x^2}}{x}dx$

p.63 (a) より $\sqrt{x^2+1} = t-x$ とおいて

$\displaystyle\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{(t^2+1)^2}{2t^2(t^2-1)}dt = \log\frac{|t-1|}{|t+1|} + \frac{t^2+1}{2t}$

長さ $\displaystyle= \Big[\log\frac{\sqrt{1+x^2}+x-1}{\sqrt{1+x^2}+x+1} + \sqrt{1+x^2}\Big]_1^a$

$= \sqrt{1+a^2} + \log(\sqrt{1+a^2}-1) - \log a - \sqrt{2} - \log(\sqrt{2}-1)$

最後のところで log の中身の分母を有理化した

(3) 長さ $\displaystyle= \int_0^{\pi/4} \sqrt{1+(-\tan x)^2}dx = \int_0^{\pi/4} \frac{1}{\cos x}dx$

ここで $\displaystyle\int \frac{1}{\cos x}dx = \int \frac{2}{1-u^2}du \quad (u = \tan\frac{x}{2}$ と置換, p. 64)

$\displaystyle= \int \Big(-\frac{1}{u-1} + \frac{1}{u+1}\Big)du = \log\Big|\frac{u+1}{u-1}\Big| = \Big[\log\Big|\frac{\cos x}{1-\sin x}\Big|\Big]$ なので

長さ $\displaystyle= \Big[\log\Big|\frac{\cos x}{1-\sin x}\Big|\Big]_0^{\pi/4} = \log(\sqrt{2}+1)$

(4) $\displaystyle y = \frac{a}{2}(e^{x/a}+e^{-x/a}) \quad (0 \le x \le b)$

$\displaystyle\int_0^b \sqrt{1+\frac{(e^{x/a}-e^{-x/a})^2}{2^2}}dx = \frac{1}{2}\int_0^b (e^{x/a}+e^{-x/a})dx$

$\displaystyle= \Big[\frac{a}{2}(e^{x/a}-e^{x/a})\Big]_0^b = \frac{a}{2}(e^{b/a}-e^{b/a})$

問題 3.4 の 2(定理 3.4.4) (2) の方が易しい

(1) $x = t\cos\dfrac{1}{t},\ y = t\sin\dfrac{1}{t}\quad(1 \leq t \leq 2)$

$$\int_1^2 \sqrt{\left(\cos\frac{1}{t}+\frac{1}{t}\sin\frac{1}{t}\right)^2+\left(\sin\frac{1}{t}-\frac{1}{t}\cos\frac{1}{t}\right)^2}dt = \int_1^2 \frac{\sqrt{1+t^2}}{t}dt$$

問題 1 の (2) と同じ

$$= \sqrt{5}+\log(\sqrt{5}-1)-\log 2-\sqrt{2}-\log(\sqrt{2}-1)$$

(2) $x = 3t^2,\ y = 3t - t^3\quad(0 \leq t \leq 2)$

$$\int_0^2 \sqrt{(6t)^2+(3-3t)^2}dt = 3\int_0^2 (t^2+1)dt$$

$$= [t^3+3t]_0^2 = 14$$

問題 3.4 の 3

(1) $x = 2t+1,\ y = 2-t-t^2$

$y = \dfrac{-x^2}{4}+\dfrac{9}{4}$ であるので x 軸との交点は $x = \pm 3$

$$\int_{-3}^3 y\,dx = 2\int_0^3 \left(\frac{-x^2}{4}+\frac{9}{4}\right)dx = 9$$

(2) $x = \sin t,\ y = t\cos t\quad(0 \leq t \leq \dfrac{\pi}{2})$

$$\int y\,dx = \int_0^{\pi/2} t\,\cos^2 t\,dt = \int_0^{\pi/2} t\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= \left[\frac{t^2}{4}\right]_0^{\pi/2}+\left[\frac{t\sin 2t}{4}\right]_0^{\pi/2}-\int_0^{\pi/2}\frac{\sin 2t}{4}dt$$

$$= \frac{\pi^2}{16}+\left[\frac{\cos 2t}{8}\right]_0^{\pi/2} = \frac{\pi^2}{16}-\frac{1}{4}$$

(3) $x = a(t-\sin t),\ y = a(1-\cos t)\quad(0 \leq t \leq 2\pi)$

$$\int y\,dx = a^2\int_0^{2\pi}(1-\cos t)^2 dt = a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt$$

$$= a^2[t-2\sin t]_0^{2\pi}+a^2\int_0^{2\pi}\frac{1+\cos 2t}{2}dt$$

$$= 2\pi a^2+a^2\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]_0^{2\pi} = 3\pi a^2$$