Asymptotic expansions of finite Hankel transforms and the surjectivity of convolution operators

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1. Invertible distributions

1. $P(D) \neq 0 \colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is surjective (Ehrenpreis, Malgrange, '56)
   (arbitrary linear PDOp with const. coeffs.)
   Convolution with $P(D)\delta(\boldsymbol{x})$.

2. Translation: $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$; $u(\boldsymbol{x}) \to u(\boldsymbol{x}-\boldsymbol{a})$ is surjective.
   Convolution with $\delta(\boldsymbol{x}-\boldsymbol{a})$.

A compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ is called

invertible if $u* \colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is surjective (Ehrenpreis).

(Only the existence of the *right* inverse is required.)

2. invertibility and slow decrease

Theorem (Ehrenpreis '60 cf. Hörmander '62, '83)

If $u \in \mathcal{E}'(\mathbb{R}^n)$, the following conditions are equivalent:

1. u is **invertible** ($u* : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is surjective)
2. The Fourier transform of u is **slowly decreasing** in the following sense:

   $\exists A > 0$ s. t.

   $$\sup\{|\hat{u}(\boldsymbol{\eta})|\,;\boldsymbol{\eta} \in \mathbb{R}^n, |\boldsymbol{\eta} - \boldsymbol{\xi}| < A\log(2+|\boldsymbol{\xi}|)\} > (A+|\boldsymbol{\xi}|)^{-A}$$

   for $\forall \boldsymbol{\xi} \in \mathbb{R}^n$.

$\hat{u}$ is allowed to have zeros.
Existence of peaks is good enough.

3. Known invertible distributions

1. $\sum_{j=1}^{J} P_j(D)\delta(\boldsymbol{x}-\boldsymbol{a}_j) \neq 0$ is invertible (probably Hörmander).
2. The delta function supported by the sphere $|\boldsymbol{x}| = r$ is invertible (Lim, 2012).
3. Its normal derivatives of arbitrary order are invertible (Okada-Y, 2021).
4. If u_1 and u_2 are invertible, so is $u_1 * u_2$.
5. Invertibility is preserved under translation and dilation.
6. If $u(\boldsymbol{x})$ and $v(\boldsymbol{y})$ are invertible, so is $u(\boldsymbol{x})v(\boldsymbol{y})$.

We want more examples and sufficient conditions.

4. Fourier transform and finite Hankel transform

We try to find radial functions with compact support that are *invertible*.

Let $f_0(s)$ be a function in a *single* variable with $\operatorname{supp} f_0 \subset [0,1]$. Set $f(\boldsymbol{x}) = f_0(|\boldsymbol{x}|), \boldsymbol{x} \in \mathbb{R}^n$(radial). Then

$$\hat{f}(\boldsymbol{\xi}) = \frac{\text{const.}}{r^{n/2-1}} \int_0^1 s^{n/2} f_0(s) J_{n/2-1}(rs)\, ds,$$
$$r = |\boldsymbol{\xi}|,\ \boldsymbol{\xi} \in \mathbb{R}^n.$$

We want to prove $\hat{f}(\boldsymbol{\xi})$ is slowly decreasing ($\Leftrightarrow f(\boldsymbol{x})$ is invertible) under certain conditions.

Estimating $\int_0^1 s^{n/2} f_0(s) J_{n/2-1}(rs)\, ds$ (function in $r = |\xi|$) is the key.

5. Asymptotic expansion of finite Hankel transforms

$\varphi(s)$ smooth in $0 < s < 1$.

Behavior of $\int_0^1 \varphi(\mathrm{s})\ J_{n/2-1}(rs)\,ds$ **is determined by the singularities at the left and right ends:** $s \to +0$ **and** $s \to 1-0$**.**

Two assumptions:

1. $\varphi(s)$ has an expansion by powers of s at the **left** end.
2. $s^{-n/2}\varphi(s)$ has an expansion by powers $1-s^2$ at the **right** end.

Three tools:

1. some kind of cut-off, contributions from the two ends are separated
2. the left end: Roderick Wong's result ('76)
3. the right end: Sonine's first finite integral, integration by parts based on the ladder operator

6. The left end $(s \to +0)$

By using the result of Wong '76, we get the following.
Let $\varphi(s)$ be $\mathcal{C}^\infty$ in $(0,1)$ and $\mathrm{Re}\,(\mu+\nu) > -1$,

Assume $\varphi^{(j)}(s) \sim \sum_{k=0}^{\infty} c_k \frac{d^j}{ds^j} s^{\mu+k} \quad (s \to +0;\ j = 0, 1, 2, \dots)$

Let $\chi_0(s)$ be a cutoff function which is 1 near the left end and set

$$K := K\left(\mu, \nu, \{c_k\}_k\right) = \left\{ k \in \mathbb{N}_0\,;\, c_k \neq 0,\ \frac{1}{2}(\mu + k - \nu - 1) \notin \mathbb{N}_0 \right\},$$

Assume $K \neq \emptyset$ and set $k_0 = \min K$. Then as $r \to \infty$

$$\int_0^1 \chi_0(s)\varphi(s) J_\nu(rs)\, ds \sim c_{k_0} \frac{\Gamma\left(\frac{1}{2}(\mu + k_0 + \nu + 1)\right) 2^{\mu+k_0}}{\Gamma\left(\frac{1}{2}(\mu + k_0 - \nu - 1)\right)} r^{-(\mu+k_0+1)}.$$

Coeffs may vanish $\Leftarrow$ poles of the Gamma function.

7.The right end $(s \to 1-0)$

Assume $s^{-n/2}\varphi(s)$ has an expansion by powers $1-s^2$ at the **right** end.
We can use Sonine's first finite integral

$$\int_0^1 s^{\nu+1}\left(1-s^2\right)^{\alpha} J_\nu(rs)\,ds = 2^{\alpha}\Gamma(\alpha+1)r^{-(\alpha+1)}J_{\nu+\alpha+1}(r).$$

The behavior of the right hand side can be calculated by using

$$J_\alpha(r) = \frac{2^{1/2}}{\pi^{1/2}} r^{-1/2}\cos\left(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(r^{-3/2}).$$

8. Main theorem (Okada-Y, SIGMA last week)

Let $\varphi(s)$ be a $\mathcal{C}^\infty$ function in $(0,1)$. Assume $\mathrm{Re}(\mu+n/2)>0$ and

$$\varphi^{(j)}(s) \sim \sum_{k=0}^{\infty} c_k \frac{d^j}{ds^j} s^{\mu+k},\, c_0 \neq 0 \quad (s \to +0;\, j=0,1,2,\dots)$$

Assume $-1 < \mathrm{Re}\,\lambda_0 < \mathrm{Re}\,\lambda_1 < \cdots < \mathrm{Re}\,\lambda_m < \mathrm{Re}\,\Lambda$, $N \leq \mathrm{Re}\,\Lambda$, $\mathrm{Re}\,\lambda_0 \leq N-1$ and

$$s^{-n/2}\varphi(s) = \sum_{k=0}^{m} a_k (1-s^2)^{\lambda_k} + (1-s^2)^{\Lambda}\psi(s^2),\, a_0 \neq 0 \quad (\text{near } s=1)$$

Here $\psi(\cdot)$ is sufficiently regular.
Then,

$$f(\boldsymbol{x}) := |\boldsymbol{x}|^{-n/2}\varphi(|\boldsymbol{x}|)\chi_{[0,1]}(|\boldsymbol{x}|) \ \ (\boldsymbol{x} \in \mathbb{R}^n;\, n \geq 2)$$

is *invertible*. Here $\chi_{[0,1]}(\cdot)$ is the indicator function of $[0,1]$.

9. Proof

Prof. Wong's result near the left end, Sonine near the right end.

Difficulty:
cutoff near $s = 1$ destroys Sonine-type integrand $(1 - s^2)^{\alpha}$.

Incomplete cutoff near $s = 1$: we consider

$$\underbrace{\sum_{k=0}^{m} a_k (1 - s^2)^{\lambda_k}}_{\text{Sonine preserved}} + \underbrace{\chi_1(s^2)}_{\text{cutoff}} (1 - s^2)^{\Lambda} \psi(s^2).$$

Hankel transform behaves like $\text{const.} \times$power of $r \times \cos.$

Usual cutoff near $s = 0$ of the difference of $\varphi(s)$ and the Sonine terms

$$\tilde{\varphi}(s) := \chi_0(s) \Big\{ \varphi(s) - \underbrace{s^{n/2} \sum_{k=0}^{m} a_k (1 - s^2)^{\lambda_k}}_{\text{Sonine}} \Big\}$$

10. Proof (continued)

$$\tilde{\varphi}(s) := \chi_0(s)\Big\{\varphi(s) - \underbrace{s^{n/2}\sum_{k=0}^{m} a_k(1-s^2)^{\lambda_k}}_{\text{Sonine}}\Big\},$$

$$\tilde{\varphi}^{(j)}(s) \sim \sum_{k=0}^{\infty} c_k \frac{d^j}{ds^j} s^{\mu+k} + \underbrace{\sum_{\ell=0}^{\infty} A_\ell \frac{d^j}{ds^j} s^{n/2+2\ell}}_{\text{additional terms}} \quad (s \to +0)$$

The additional terms do not contribute to the Wong expansion, because of the poles of the Gamma function.
SLOW DECREASE in any of the three cases

1. Contribution from the left end (power, Wong) is dominant.
2. That from the right (power $\times$ oscillation, Sonine) is dominant.
3. They are of the same order.

Happy birthday, Professor Wong!