Local and global analyticity for a generalized Camassa-Holm system

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1. (single) Camassa-Holm equation

Camassa-Holm equation $u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}$ or

$$u_t + uu_x + \partial_x (1 - \partial_x^2)^{-1} \left[u^2 + \frac{1}{2} u_x^2 \right] = 0 \text{ on } \mathbb{R},$$

where

$$(1 - \partial_x^2)^{-1} \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (1 + \xi^2)^{-1} \hat{\varphi}(\xi) \, d\xi.$$

Shallow water wave, bi-Hamiltonian structure, integrability, ...

variations:
Periodic ($x \in S^1$),
μ (involves mean value on $S^1 \ni x$), Khesin-Lenells-Misiolek.
System

2. Global analytic solution (Barostichi-Himonas-Petronilho, JDE 2017)

Very roughly speaking, if the initial value is holomorphic in a strip $\supset \mathbb{R}$ in $\mathbb{C}$ and is square-integrable, then the solution to IVP (for a generalize CH) exists globally in time and is analytic.

Methods:

Introduce suitable function spaces

Local analytic solution: abstract Cauchy-Kowalevsky. Scales of Banach spaces.

Time-global H^s solution

Analyticity in x for any t: Method by Tosio Kato and Kyuya Masuda

Analyticity in (t, x): Komatsu or Kotake-Narashimhan (BHP quotes a book by Rodino.)

3. CH system of R. M. Chen-Y. Liu

Chen-Liu (IMRN 2011)

$$\begin{cases} u_t - u_{txx} - \alpha u_x + 3uu_x - \beta(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1}$$

Here it is assumed that $u \to 0$ and $\rho \to 1$ hold as $|x| \to \infty$.
Set $v = \rho - 1 \to 0$.
(1) is equivalent to

$$\begin{cases} u_t + \beta uu_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0. \end{cases} \tag{2}$$

with $u \to 0$, $v \to 0$ as $|x| \to \infty$.

4. Formulation of IVPs

The CH system of Chen-Liu

$$\begin{cases} u_t + \beta u u_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0. \end{cases}$$

with $u \to 0$, $v \to 0$ involves the ΨDO $(1-\partial_x^2)^{-1}$.
So research must be GLOBAL in x.

It can be solved LOCALLY or GLOBALLY in t.
Solutions in a suitable space of functions on $\mathbb{R}_x$.

5. Known result: time-global solvability in H^s

Theorem (Chen-Liu 2011)

Assume $0 < \beta < 2$, $s > 3/2$. *If* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *and* $\inf_{x \in \mathbb{R}} v_0(x) > -1$, *then the IVP for*

$$\begin{cases} u_t + \beta u u_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0 \end{cases}$$

with $u(0,x) = u_0$, $v(0,x) = v_0$ *has a unique solution* (u,v) *in* $\mathcal{C}([0,\infty), H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap \mathcal{C}^1([0,\infty), H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$.

6. Main result: global analytic solution

If the initial data are analytic, then the solution is analytic globally in both t and x. **(μ-case is by Y., 2020)**

For $r > 0$, set $S(r) = \{x + iy \in \mathbb{C}; |y| < r\}$ and

$$A(r) = \{f : \mathbb{R} \to \mathbb{R}; f(z) \text{ can be analytically continued to } S(r)\} \cap \left\{ f \in L^2_{x,y}(S(r')) \text{ for all } 0 < r' < r \right\}.$$

Theorem (**Global analyticity [Funk. Ekvac. 2023]**)

Assume $0 < \beta < 2$ *and* $\inf_{x\in\mathbb{R}} v_0(x) > -1$.
If $u_0, v_0 \in A(r_0)$ *for some* $r_0 > 0$,
then the solution (u, v) *is analytic in* t, x. *It belongs to* $\oplus^2 \mathcal{C}^\omega([0,\infty)_t \times \mathbb{R}_x)$.

7. time-local and global analyticity

IVP for the CH system with analytic initial value (with some technical assumptions).
$\Rightarrow$ Unique existence of a global-in-time analytic solution

Ref: (generalized) CH, Barostichi-Himonas-Petronilho 2017

WHAT REMAINS TO BE PROVED (solvability in H^s is known):

1. local analyticity in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument
2. analyticity in x ($t > 0$ fixed)
   $\leftarrow$ Kato-Masuda theory. The most difficult part.
3. global analyticity in t

8. $A(r)$ (Fréchet) and $E_{\delta,s}$ (Banach)

Following BHP (with some generalization and a modified notation), we introduce

$$\|f\|_{(\delta,s)} = \sup_{k\geq 0} \frac{\delta^k (k+1)^2 \|f^{(k)}\|_s}{k!} \ (0 < \delta \leq 1, s \geq 2).$$

and the Banach space $E_{\delta,s}$ by

$$E_{\delta,s} = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}); \|f\|_{(\delta,s)} < \infty \right\}.$$

$E_{\delta,s}$ is closed under multiplication.
$E_{\delta,s}$ is continuously embedded in $A(\delta)$.
Conversely, if $\delta < r/e$ then $A(r)$ is continuously embedded in $E_{\delta,s}$.

9. Continuity of operations on $E_{\delta,s}$

If $0 < \delta \leq 1$, $s \geq 2$, then

$$\|uv\|_{(\delta,s)} \leq \text{const.} \|u\|_{(\delta,s)} \|v\|_{(\delta,s)}.$$

If $0 < \delta' < \delta \leq 1$, we have

$$\|\partial_x u\|_{(\delta',s)} \leq \frac{1}{\delta - \delta'} \|u\|_{(\delta,s)},$$

$$\|\partial_x u\|_{(\delta,s)} \leq \|u\|_{(\delta,s+1)},$$

$$\|(1-\partial_x^2)^{-1} \partial_x^p u\|_{(\delta,s)} \leq \|u\|_{(\delta,s)} \ (p = 0, 1, 2),$$

$$\|(1-\partial_x^2)^{-1} \partial_x u\|_{(\delta',s)} \leq \frac{\|u\|_{(\delta,s)}}{\delta - \delta'},$$

$$\|(1-\partial_x^2)^{-1} u\|_{(\delta,s+2)} = \|u\|_{(\delta,s)} \ (p = 0, 1, 2),$$

$$\|(1-\partial_x^2)^{-1} \partial_x u\|_{(\delta',s+1)} \leq \frac{1}{\delta - \delta'} \|u\|_{(\delta,s)}.$$

10. time-local analytic IVP for CH system

Theorem

Let $0 < \Delta \leq 1, s \geq 2$. *If* $(u_0, v_0) \in \oplus^2 E_{\Delta,s+1}$, *then there exists* $T_\Delta > 0$ *such that the IVP the CH system has a unique holomorphic solution valued in* $\oplus^2 E_{\Delta d,s+1}$ *in the disk* $D(0, T_\Delta(1-d))$ *for every* $d \in]0,1[$. *(t is near 0)*

Method: abstract Cauchy-Kowalevsky. Scales of Banach spaces. (Ovsyannikov, Yamanaka, Trèves)

Ref: CH and similar equations, Barostichi-Himonas-Petronilho 2016

We used $\|\cdot\|_{(\delta,s)}$ to prove local analyticity in t (small).

11. New norm $\|\bullet\|_{\sigma,2}$

Next, we want to show analyticity in x (for fixed $t \in \mathbb{R}$).
Following Kato-Masuda (1986), set

$$\|f\|_{\sigma,2}^2 = \sum_{j=0}^{\infty} \frac{e^{2j\sigma}}{j!^2} \|f^{(j)}\|_2^2.$$

Do not confuse $\|\bullet\|_{\sigma,2}$ with $\|\bullet\|_{(\delta,s)}$.

$\|\bullet\|_{\sigma,2}$ is useful in the study of analytic functions:

If $f \in A(r)$, then $\|f\|_{\sigma,2} < \infty$. (Here $\sigma < \log r$)

If $\|f\|_{\sigma,2} < \infty$ for any $\sigma < \log r$, then $f \in A(r)$.

We employ $\|\bullet\|_{\sigma,2}$ to prove analyticity in x
for an arbitrarily large (fixed) t.

12. Regularity theorem by Kato and Masuda: outline

Consider the equation

$$\frac{dU}{dt} = F(U),\, U(0) = U_0.$$

Here F is typically a (nonlinear) continuous mapping from a Banach space to another.
Kato-Masuda theorem gives some sufficient condition for the regularity of $U(t), t > 0$.
If U_0 is regular to some extent, then so is $U(t), t > 0$.

Let $\{\Phi_\sigma; -\infty < \sigma < \infty\}$ be a family of functions related to norms on Banach spaces. (Liapunov family).

Φ_σ is a measure of regularity.
$\Phi_\sigma(U(t))$ can be estimated in terms of U_0.

13. Regularity theorem by Kato-Masuda: formulation

X, Z: Banach spaces and Z is a dense subspace of X.
F : continuous mapping from Z to X.
$\{\Phi_\sigma; -\infty < \sigma < \infty\}$: a family of real-valued functions on Z.
Assume

$$|\langle F(U), D\Phi_s(U)\rangle| \le K\Phi_s(U) + L\Phi_s(U)^{1/2}\partial_s\Phi_s(U) + M\partial_s\Phi_s(U).$$

D: Frechét derivative
$\langle\cdot,\cdot\rangle$ (no subscript) : the pairing of X and $\mathcal{L}(X;\mathbb{R})$.
If $dU/dt = F(U)$, $U(0) = U_0$, then for functions $s(t), r(t)$ depending on U_0 we have

$$\Phi_{s(t)}(U(t)) \le r(t),\ t \in [0, T].$$

If U_0 is regular to some extent, then so is $U(t), t > 0$.

14. Liapunov family: the case of the CH system

The system is asymmetric in $(u, v)$$\Rightarrow$asymmetric Liapunov family

Set $X = \oplus^2 H^{m+2}, \quad Z = \oplus^2 H^{m+4}$,

$$\Phi_{\sigma,m}(u,v) = \Phi^{(1)}_{\sigma,m}(u) + \Phi^{(2)}_{\sigma,m}(v),$$

$$\Phi^{(1)}_{\sigma,m}(u) = \frac{1}{2}\sum_{j=1}^{m+1} \frac{1}{j!^2} e^{2(j-1)\sigma} \frac{\|u^{(j)}\|_2^2}{2},$$

$$\Phi^{(2)}_{\sigma,m}(v) = \frac{1}{2}\sum_{j=0}^{m} \frac{1}{j!^2} e^{2j\sigma} \frac{\|v^{(j)}\|_2^2}{2}.$$

Then

$$\|u\|^2_{\sigma,2} = \|u\|_2^2 + 2\lim_{m\to\infty} e^{2\sigma}\Phi^{(1)}_{\sigma,m}(u),$$

$$\|v\|^2_{\sigma,2} = \lim_{m\to\infty} 2\Phi^{(2)}_{\sigma,m}(v)$$

and if they are finite, u and v are analytic in x.

We want to get bounds on $\Phi_{\sigma,m}(u,v)$ by using KM theory.

15. Rewriting the system

$$F(u,v) = (F_1(u,v), F_2(u,v)),$$
$$F_1(u,v) = -\beta u u_x - (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2\right],$$
$$F_2(u,v) = -u_x - (uv)_x.$$

Our CH system is

$$\frac{d(u,v)}{dt} = F(u,v)$$

and this is how the Kato-Masuda theory is applied.

16. Kato-Masuda and the CH system

F is a continuous mapping from $\oplus^2 H^{m+4}$ to $\oplus^2 H^{m+2}$.
There exist positive constants $K_1, K_2, L_1, L_2, M_1, M_2, M_3$ independent of u, v and σ such that we have

$$
\begin{aligned}
&|\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle| \\
&\le [K_1 + K_2\|(u,v)\|_3]\,\Phi_{\sigma,m}(u,v) \\
&\quad + (L_1 + L_2 e^{\sigma})\,\Phi_{\sigma,m}(u,v)^{1/2}\partial_\sigma\Phi_{\sigma,m}(u,v) \\
&\quad + \left[M_1 + (M_2 + M_3 e^{2\sigma})\|(u,v)\|_3\right]\partial_\sigma\Phi_{\sigma,m}(u,v)
\end{aligned}
$$

for $(u, v) \in \oplus^2 H^{m+4}$.
Kato-Masuda theory works for $d(u, v)/dt = F(u, v)$ [CH system].
$\Rightarrow$ Bounds on $\Phi_{\sigma,m}(u(t), v(t))$ and regularity of $(u(t), v(t))$.
$m \to \infty$ and $u(t)$ and $v(t)$ are analytic in x for any $t > 0$.

17. Estimating $\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle$

$$\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle$$
$$= \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2}\langle u^{(j)}, \partial_x^j F_1(u,v)\rangle_2 + \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2}\langle v^{(j)}, \partial_x^j F_2(u,v)\rangle_2,$$

The bracket on the left-hand side is the pairing of $\oplus^2 H^{m+2}$ and its dual $(\oplus^2 H^{m+2})^* \simeq \oplus^2 H^{m+2}$.
$\langle\cdot,\cdot\rangle_2$ is the inner product of H^2.
Estimates by using

$\|fg\|_2 \leq 8(\|f\|_2\|g\|_1 + \|f\|_1\|g\|_2)$ (Kato-Ponce).

H^2, H^1 norms in RHS. Better than $\|fg\|_2 \leq \text{const.}\|f\|_2\|g\|_2$.

18. Estimating $\sum_{j=1}^{m+1} j!^{-2} e^{2(j-1)\sigma} \langle u^{(j)}, \partial_x^j (uu_x)\rangle_2$

$\displaystyle\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \partial_x^j F_1\rangle_2$ involves

$$Q_j = \sum_{\ell=1}^{j} \binom{j}{\ell} \langle u^{(j)}, u^{(\ell)} u^{(j-\ell+1)}\rangle_2. \quad \text{(degree3)}$$

Apply Schwarz and get $\|u^{(j)}\|_2 \|u^{(\ell)} u^{(j-\ell+1)}\|_2$. By Kato-Ponce,

$$\begin{aligned}\|u^{(\ell)} u^{(j-\ell+1)}\|_2 &\le 8\left(\|u^{(\ell)}\|_2 \|u^{(j-\ell+1)}\|_1 + \|u^{(\ell)}\|_1 \|u^{(j-\ell+1)}\|_2\right) \\ &\le 8\left(\|u^{(\ell)}\|_2 \|u^{(j-\ell)}\|_2 + \|u^{(\ell-1)}\|_2 \|u^{(j-\ell+1)}\|_2\right).\end{aligned}$$

$$\left|\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \partial_x^j (uu_x)\rangle_2\right|$$

$$\le 96\|u\|_3 \Phi_{\sigma,m}(u,v) + \left(16\|u\|_3 + \frac{32\pi}{\sqrt{3}} e^{\sigma} \sqrt{\Phi_{\sigma,m}(u,v)}\right) \partial_\sigma \Phi_{\sigma,m}(u,v).$$

19. Final part of the proof of the main result

1. analyticity in t and x, local in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument

2. analyticity in x (arbitrarily large fixed $t > 0$)
   $\leftarrow$ Kato-Masuda, just completed

3. global analyticity in t $\leftarrow$ combination of 1 and 2

4. analyticity of $\mathbb{R}_t \to A(r)$(function space in x, infininite dim.)
$\Rightarrow$ analyticity of $\mathbb{R}^2_{t,x} \to \mathbb{R}$

$t \mapsto u(t, \cdot)$ is analytic.

$$\left\| \partial_x^k \partial_t^j u \right\|_{L^2(\mathbb{R}\times[-T,T])} \leq \sqrt{T} C^{j+k+1} (j+k)!.$$

u is analytic in (t, x) by Komatsu (1960) or Kotake-Narashimhan (1961).

Prof. Honda and Prof. Okada, congratulations on your 60th birthdays!

Osaka Umeda Seminar on Functional Equations and Special Functions
Kwansei Gakuin University, Umeda Campus, Oct 12 (Sat.)
Speakers: Nobukawa, Tsuchimi, Nakamura, Yamane