Asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

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1. Riemann-Hilbert problem (RHP)

BOUNDARY VALUE PROBLEM IN THE COMPLEX PLANE

Γ: oriented contour (the left-hand is the $+$ side).

$m(z)$: unknown matrix, holomorphic in $\mathbb{C} \setminus \Gamma$

Examples:
1. $\Gamma = \mathbb{R}$, $m(z)$ holo. in $\pm \mathrm{Im}\, z > 0$.
2. $\Gamma = \{|z| = 1\}$, $m(z)$: holo. in $|z| \neq 1$.

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2. $\Gamma = \{|z| = 1\}$, $m(z)$: holo. in $|z| \neq 1$.

m_+, m_-: boundary values on Γ from the $\pm$ sides

$$\boxed{\boxed{\textit{RHP:}\ \ m_+ = m_- v \text{ on } \Gamma}} \quad (v : \textbf{the jump matrix})$$

We often neglect to mention the normalization condition $m(z) \to I$ as $z \to \infty$.

2. RHPs behave like integrals

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contour deformation

New contour, unknown, jump matrix.
The original RHP $\Leftrightarrow$ new RHP.

continuity

The mapping $v \mapsto m$ is continuous.

deletion of a part of the contour

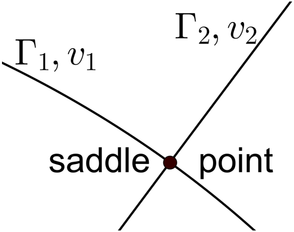
1. If $v = I$ on $\hat{\Gamma} \subset \Gamma$ (no jump there),
$$m[\text{original}] = m[\text{with } \hat{\Gamma} \text{ deleted}]$$
2. If $v \approx I$ on $\hat{\Gamma}$, $m[\text{original}] \approx m[\text{with } \hat{\Gamma} \text{ deleted}]$

3. Nonlinear steepest descent (Deift-Zhou '93)

An RHP $m_+ = m_- v$ behaves like an integral.
An analogue of the method of steepest descent is possible.

Deform Γ if necessary and we asssume:
$v = v_j$ on $\Gamma_j \subset \{\mathrm{Im}\,(-1)^{j-1}\psi > 0\}\,(j = 1, 2)$,
v_1 involves $\underline{\exp(it\psi) \to 0}$, v_2 involves $\underline{\exp(-it\psi) \to 0}$

$\underline{v_j \to I}$ on $\Gamma_j \setminus \{$saddle point$\}$.



$\underline{m(z) \text{ is almost determined by } v(z)\,(z \text{ near the saddle point}).}$

4. Inverse scattering for NLS and RHP

$$\boxed{iu_t + u_{xx} - 2|u|^2 u = 0 \cdots (\mathrm{NLS})}$$

$r(z,t)$: reflection coefficient

$$v_1(z) := \begin{bmatrix} 1 - |r(z,0)|^2 & -e^{-2it\psi_1}\overline{r(z,0)} \\ e^{2it\psi_1} r(z,0) & 1 \end{bmatrix}, \quad \psi_1 := 2z^2 + \frac{xz}{t}$$

$$\begin{aligned} m_+(z) &= m_-(z) v_1(z) \quad \text{on } \mathbb{R}, \\ m(z) &\to I \ (z \to \infty) \end{aligned}$$

Reconstruction formula ←INVERSE PROBLEM!

$$u(x,t) = 2i \lim_{z\to\infty} z\, m(z;x,t)_{12} \ \text{(Ablowitz-Clarkson)}$$

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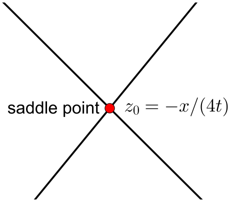
$$\boxed{u(x,0) \underset{x}{\mapsto} r(z) = r(z,0) \underset{t}{\mapsto} r(z,t) \mapsto m \mapsto u(x,t)}$$

5. Asymptotics of NLS

1. Zakhalov-Manakov '76: formal calculation
2. Deift-Its-Zhou '93: proof by $\boxed{\textit{nonlinear steepest descent}}$

RHP involving $\exp(it\psi_1)$

$\psi_1 = 2tz^2 + xz/t;\quad z_0 = -x/(4t)$ is the only saddle point



contour deformation: $\mathbb{R} \to$ cross as above

$$\boxed{u(x,t) \sim \alpha(z_0) t^{-1/2} \exp\left(4itz_0^2 - i\nu(z_0)\log 8t\right)}$$

6. Integrable Discrete NLS (IDNLS)

Ablowitz-Ladik ('75) introduced

$$i\frac{d}{dt}R_n+(R_{n+1}-2R_n+R_{n-1})-|R_n|^2(R_{n+1}+R_{n-1})=0 \quad \text{(IDNLS)}$$

cf. nonlinear optical waveguides, melting chrystal, ...

R_n is asymptotically (Y. 2014, 2015)

1. $|n|/t < 2$

   Sum of two terms, each being $\underline{t^{-1/2}\times\text{(oscillatory factor)}}$

2. $|n|/t \approx 2$

   $\underline{t^{-1/3}\times\text{(oscillatory factor)}}$

   coefficient written in terms of a sol. of the Painlevé II.

   $$u'' - su(s) - 2u^3(s) = 0$$

3. $|n|/t > 2$

   $O(n^{-j})$ as $n \to \infty$

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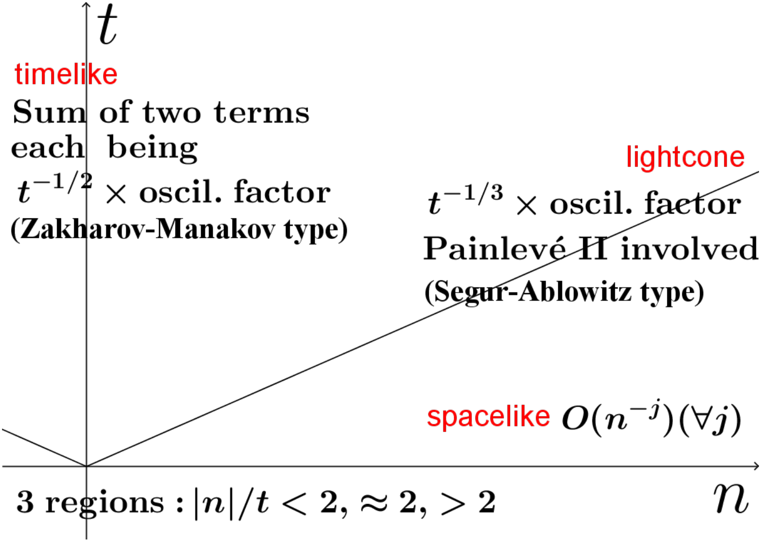
cf. formal calculation by Novokshënov about the focusing, solitonless case

7. Asymptotics: three regions



8. IDNLS and its Lax pair

$$i\frac{d}{dt}R_n+(R_{n+1}-2R_n+R_{n-1})-|R_n|^2(R_{n+1}+R_{n-1})=0 \quad \text{(IDNLS)}$$

. . . .

n- and t-parts

$$X_{n+1}=\begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$$

$$\frac{d}{dt}X_n=\Bigg[\text{a complicated matrix}\Bigg]X_n$$

(IDNLS) is the compatibility condition.

9. Reflection coefficient

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$$

Ψ_n: holo. sol. in $\boxed{|z| > 1}$, continuous in $|z| \geq 1$,

Ψ_n^*: holo. sol. in $\boxed{|z| < 1}$, continuous in $|z| \leq 1$,

$$\Psi_n \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Psi_n^* \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{as } n \to \infty.$$

The $\boxed{\text{reflection coefficient } r}$ is defined by :

$$\underbrace{r\Psi_n}_{\text{reflection}} + \underbrace{\Psi_n^*}_{\text{incidence}} \sim \mathrm{const.} z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (n \to -\infty).$$

$r(z,t) = r(z)\exp\left(it(z - z^{-1})^2\right)$, where $r(z) = r(z,0)$.

10. RHP

$$m_+(z) = m_-(z)v_2(z) \text{ on } \underline{|z| = 1},$$

$$m(z) \to I \text{ as } z \to \infty,$$

$$v_2(z) = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_2}\bar{r}(z) \\ e^{2it\psi_2}r(z) & 1 \end{bmatrix} \text{ jump matrix}$$

$$\psi_2 = \frac{1}{2}(z - z^{-1})^2 + \frac{in}{t}\log z$$

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Reconstruction formula $R_n(t) = -\frac{d}{dz}m(z)_{21}\Big|_{z=0}$

$\boxed{\text{RHP gives } \{R_n\}.}$

ψ_2 has four saddle points. Their geometry (relative to $|z| = 1$) determines the asymptotic behavior of R_n.

Thank you very much!

太感谢了!