Asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

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1. Riemann-Hilbert problem (RHP)

BOUNDARY VALUE PROBLEM IN THE COMPLEX PLANE

\[ \Gamma: \text{ oriented contour (the left-hand is the + side).} \]
\[ m(z): \text{ unknown matrix, holomorphic in } \mathbb{C} \setminus \Gamma \]

\textbf{Examples:}
1. \( \Gamma = \mathbb{R} \), \( m(z) \) holo. in \( \pm \text{Im} \, z > 0 \).
2. \( \Gamma = \{ |z| = 1 \} \), \( m(z) \): holo. in \( |z| \neq 1 \).
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2. $\Gamma = \{|z| = 1\}$, $m(z)$: holo. in $|z| \neq 1$.

$m_+, m_-$: boundary values on $\Gamma$ from the ± sides

$\textbf{RHP: } m_+ = m_- v$ on $\Gamma$ (v: the jump matrix)

We often neglect to mention the normalization condition

$m(z) \to I$ as $z \to \infty$. 
2. RHPs behave like integrals

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**RHP:** \( m_+ = m_- v \) on \( \Gamma \)

- **contour deformation**
  - New contour, unknown, jump matrix.
  - The original RHP \( \iff \) new RHP.

- **continuity**
  - The mapping \( v \mapsto m \) is continuous.

- **deletion of a part of the contour**
  1. If \( v = I \) on \( \hat{\Gamma} \subset \Gamma \) (no jump there),
     \[
     m[\text{original}] = m[\text{with } \hat{\Gamma} \text{ deleted}]
     \]
  2. If \( v \approx I \) on \( \hat{\Gamma} \), \( m[\text{original}] \approx m[\text{with } \hat{\Gamma} \text{ deleted}] \)
3. Nonlinear steepest descent (Deift-Zhou ’93)

An RHP $m_+ = m_- v$ behaves like an integral. An analogue of the method of steepest descent is possible. Deform $\Gamma$ if necessary and we assume:

$v = v_j$ on $\Gamma_j \subset \{ \text{Im } (-1)^{j-1} \psi > 0 \}$ ($j = 1, 2$),

$v_1$ involves $\exp(it\psi) \to 0$, $v_2$ involves $\exp(-it\psi) \to 0$

$v_j \to I$ on $\Gamma_j \setminus \{ \text{saddle point} \}$.

$m(z)$ is almost determined by $v(z)$ ($z$ near the saddle point).
4. Inverse scattering for NLS and RHP

\[ iu_t + u_{xx} - 2|u|^2u = 0 \cdots \text{(NLS)} \]

\( r(z, t) \): reflection coefficient

\[ v_1(z) := \begin{bmatrix} 1 - |r(z, 0)|^2 & -e^{-2it\psi_1}r(z, 0) \\ e^{2it\psi_1}r(z, 0) & 1 \end{bmatrix}, \quad \psi_1 := 2z^2 + \frac{xz}{t} \]

\[ m_+(z) = m_-(z)v_1(z) \quad \text{on } \mathbb{R}, \]

\[ m(z) \to I \text{ as } z \to \infty \]

Reconstruction formula \( \leftarrow \text{INVERSE PROBLEM!} \)

\[ u(x, t) = 2i \lim_{z \to \infty} z m(z; x, t)_{12} \text{ (Ablowitz-Clarkson)} \]
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\[
u(x,0) \leftrightarrow r(z) = r(z,0) \leftrightarrow r(z, t) \leftrightarrow m \leftrightarrow u(x, t)
\]
5. Asymptotics of NLS

1. Zakhalov-Manakov '76: formal calculation
2. Deift-Its-Zhou '93: proof by nonlinear steepest descent

RHP involving \( \exp(it\psi_1) \)

\[
\psi_1 = 2tz^2 + xz/t; \quad z_0 = -x/(4t)
\]

is the only saddle point

\[
u(x, t) \sim \alpha(z_0)t^{-1/2} \exp(4itz_0^2 - iv(z_0)\log 8t)
\]
6. Integrable Discrete NLS (IDNLS)

Ablowitz-Ladik ('75) introduced

\[ i \frac{d}{dt} R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2 (R_{n+1} + R_{n-1}) = 0 \]  (IDNLS)

cf. nonlinear optical waveguides, melting chrystal, ...

\( R_n \) is asymptotically (Y. 2014, 2015)

1. \(|n|/t < 2\)
   Sum of two terms, each being \( t^{-1/2} \times \) (oscillatory factor)

2. \(|n|/t \approx 2\)
   \( t^{-1/3} \times \) (oscillatory factor)
   coefficient written in terms of a sol. of the Painlevé II.
   \[ u'' - su(s) - 2u^3(s) = 0 \]

3. \(|n|/t > 2\)
   \( O(n^{-j}) \) as \( n \to \infty \)

cf. formal calculation by Novokshënov about the focusing, solitonless case
7. Asymptotics: three regions

sum of two terms each being
\( t^{-1/2} \times \text{oscil. factor} \)
(Zakharov-Manakov type)

3 regions: \(|n|/t < 2, \approx 2, > 2\)

- Timelike
- Lightcone
- Spacelike \( O(n^{-j}) (\forall j) \)
8. IDNLS and its Lax pair

\[ i \frac{d}{dt} R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2 (R_{n+1} + R_{n-1}) = 0 \] (IDNLS)

\[ X_{n+1} = \begin{bmatrix} z & \bar{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n \]

\[ \frac{d}{dt} X_n = \begin{bmatrix} \text{a complicated matrix} \end{bmatrix} X_n \]

(IDNLS) is the compatibility condition.
9. Reflection coefficient

\[ X_{n+1} = \begin{bmatrix} z & R_n \\ R_n & z^{-1} \end{bmatrix} X_n \]

\( \Psi_n \): holo. sol. in \(|z| > 1\), continuous in \(|z| \geq 1\),

\( \Psi^*_n \): holo. sol. in \(|z| < 1\), continuous in \(|z| \leq 1\),

\[ \Psi_n \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Psi^*_n \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{as } n \to \infty. \]

The \textbf{reflection coefficient} \( r \) is defined by:

\[ r \Psi_n + \Psi^*_n \sim \text{const.} z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (n \to -\infty). \]

\[ r(z, t) = r(z) \exp \left( it(z - z^{-1})^2 \right), \text{ where } r(z) = r(z, 0). \]
10. RHP

\[ m_+(z) = m_-(z) v_2(z) \text{ on } |z| = 1, \]
\[ m(z) \to I \text{ as } z \to \infty, \]
\[ v_2(z) = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_2} \bar{r}(z) \\ e^{2it\psi_2}r(z) & 1 \end{bmatrix} \text{ jump matrix} \]
\[ \psi_2 = \frac{1}{2} (z - z^{-1})^2 + \frac{in}{t} \log z \]
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Reconstruction formula \( R_n(t) = -\frac{d}{dz}m(z)_{21} \bigg|_{z=0} \)

RHP gives \( \{R_n\} \).

\( \psi_2 \) has four saddle points. Their geometry (relative to \( |z| = 1 \)) determines the asymptotic behavior of \( R_n \).
Thank you very much!
太感谢了!