$\mathbb{A}^1_*$-FIBRATIONS ON AFFINE THREEFOLDS

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ABSTRACT. The main theme of the present article is $\mathbb{A}^1_*$-fibrations defined on affine threefolds. The difference between $\mathbb{A}^1_*$-fibration and the quotient morphism by a G_m-action is more essential than in the case of an $\mathbb{A}^1$-fibration and the quotient morphism by a G_a-action. We consider necessary (and partly sufficient) conditions under which a given $\mathbb{A}^1_*$-fibration becomes the quotient morphism by a G_m-action. Then we consider flat $\mathbb{A}^1_*$-fibrations which are expected to be surjective, but this turns out to be not the case by an example of Winkelmann [47]. This example gives also a quasi-finite endomorphism of $\mathbb{A}^2$ which is not surjective [14]. We consider then the structure of a smooth affine threefold which has a flat $\mathbb{A}^1$-fibration or a flat $\mathbb{A}^1_*$-fibration. More precisely, we consider affine threefolds with the additional condition that they are contractible.

INTRODUCTION

In affine algebraic geometry, our knowledge on affine algebraic surfaces is fairly rich with various methods of studying them. Meanwhile, knowledge of affine threefolds is very limited. It is partly because strong geometric approaches are not available or still under development.

A possible geometric approach is to limit ourselves to the case where the affine threefolds in consideration have fibrations by surfaces or curves, say the affine plane $\mathbb{A}^2$ or the affine line $\mathbb{A}^1$ or have group actions which give the quotient morphisms. To be more concrete, $\mathbb{A}^2$-fibrations were observed to give characterizations of the affine 3-space $\mathbb{A}^3$ (see [35, 37, 20]). Meanwhile, the quotient morphism $q : Y \to Y//G_a$ for an affine threefold has been considered by many people including

P. Bonnet, Sh. Kaliman, D. Finston, J.K. Deveney et al. See [9] for the references.

In [9], some of the present authors considered $\mathbb{A}^1$-fibrations and quotient morphisms for an action of G_a. There, a main point is that an $\mathbb{A}^1$-fibration is factored by a quotient morphism by G_a and the study is essentially reduced to the quotient morphism by G_a. In this article, we consider $\mathbb{A}^1_*$-fibrations and quotient morphisms by G_m. Contrary to the case of $\mathbb{A}^1$-fibrations, $\mathbb{A}^1_*$-fibrations are not necessarily factored by a quotient morphism by G_m. The possibility to factor depends on the nature of the singular fibers of the $\mathbb{A}^1_*$-fibration (see Theorem 2.20). The quotient morphism by G_m provides us a fertile ground for research, and the $\mathbb{A}^1_*$-fibrations seem to be even more fertile, but mysterious. For example, singular fibers of $\mathbb{A}^1_*$-fibration are fully described in the surface case (see Lemma 2.2) but not completely in the case of threefolds, and the locus of singular fibers is shown to be a closed set with the unmixedness condition on singular fibers (cf. Lemma 2.11).

Our primary purpose in this article is to develop a theory of dealing with $\mathbb{A}^1_*$-fibrations by combining algebraic geometry with algebraic topology and commutative algebra and to apply it to elucidate the structure of such objects as homology (or contractible) threefolds. Thus the article contains known results as well as what we hope to be original ones.

Here we comment on the notation $\mathbb{A}^1_*$ for which the notation $\mathbb{C}^*$ is in more common use. But $\mathbb{C}^*$ is used to mean, for example, that an affine scheme Spec A has only constant invertible functions, i.e., $A^* = \mathbb{C}^*$. It is better to distinguish the curve $\mathbb{C}^*$ from the multiplicative group $\mathbb{C}^*$. Hence we use $\mathbb{A}^1_*$ for the curve and G_m for the group.

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Finally the authors appreciate various comments by the referees. As pointed out by one of them, $\mathbb{Q}$-homology threefold might as well be called $\mathbb{Q}$-homology 3-space.

1. Preliminaries

We summarize various results which we make use of in the subsequent arguments. The ground field is always the complex number field $\mathbb{C}$. A *homology n-fold* is a smooth affine algebraic variety X of dimension n such that $H_i(X; \mathbb{Z}) = 0$ for every $i > 0$. If $H_i(X; \mathbb{Q}) = 0$ for every $i > 0$ instead, we call X a $\mathbb{Q}$-*homology n-fold*.

Lemma 1.1. *Let $X = \operatorname{Spec} A$ be a homology n-fold. Let (V, D) be a pair of a smooth projective variety V and a reduced effective divisor D on V such that X is a Zariski open set of V with $D = V - X$ and D is a divisor with simple normal crossings. Then the following assertions hold.*

 (1) *X is factorial and $A^* = \mathbb{C}^*$.*
 (2) *$H^1(V, \mathcal{O}_V) = H^2(V, \mathcal{O}_V) = 0$.*

Proof. Similar considerations can be found in [6]. To simplify the arguments, we treat the case $n = 3$. Consider the exact sequence of $\mathbb{Z}$-cohomology groups for the pair (V, D),

$$
\begin{array}{lllllll}
0 & \to & H^0(V, D) & \to & H^0(V) & \to & H^0(D) & \to & H^1(V, D) \\
& \to & H^1(V) & \to & H^1(D) & \to & H^2(V, D) & \to & H^2(V) \\
& \to & H^2(D) & \to & H^3(V, D) & \to & H^3(V) & \to & H^3(D) \\
& \to & H^4(V, D) & \to & H^4(V) & \to & H^4(D) & \to & H^5(V, D) \\
& \to & H^5(V) & \to & H^5(D) & & & &
\end{array}
$$

By Lefschetz duality, we have $H^i(V, D) \cong H_{6-i}(X)$ for $0 \le i \le 6$. Since X is a homology manifold, $H_i(X) = 0$ for $1 \le i \le 3$ and since X is affine, $H_i(X) = 0$ for $4 \le i \le 6$ (see [32, Theorem 7.1]). Hence we obtain the isomorphism $H^i(V) \cong H^i(D)$ for $0 \le i < 6$. Since $H^5(D) = 0$ as $\dim D = 2$, we have $H^5(V; \mathbb{Z}) = 0$. By Poincaré duality, we have $H_1(V; \mathbb{Z}) = 0$. Then the universal coefficient theorem [46, Theorem 3, p.243] implies that $H^1(V; \mathbb{Z}) = 0$, and hence $H^1(V; \mathbb{C}) = 0$. The Hodge decomposition then implies that $H^1(V, \mathcal{O}_V) = 0$. Now consider an exact sequence

$$
0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_V \overset{\exp(2\pi i\ )}{\longrightarrow} \mathcal{O}_V^* \longrightarrow 0
$$

and the associated exact sequence

$$
(1) \qquad H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^*) \to H^2(V; \mathbb{Z}) \to H^2(V, \mathcal{O}_V)
$$

On the other hand, consider the isomorphism $H^4(V; \mathbb{Z}) \cong H^4(D; \mathbb{Z})$. By the Mayer-Vietoris exact sequence, it follows that $H^4(D; \mathbb{Z})$ is a free abelian group generated by the classes of the irreducible components of

D. The Poincaré duality and the universal coefficients theorem implies

$$H^4(V;\mathbb{Z}) \cong H_2(V;\mathbb{Z}) \cong H^2(V;\mathbb{Z}),$$

where we note that $H_1(V;\mathbb{Z}) = 0$. Hence we know that $H^2(V;\mathbb{Z})$ is a free abelian group generated by the first Chern classes of the irreducible components of D. Since all these classes are algebraic, we conclude that $H^2(V;\mathbb{Z}) = H^{1,1}(V) \cap H^2(V;\mathbb{C})$. Hence $H^2(V,\mathcal{O}_V) = 0$.

Now, by (1) above, we know that

$$\mathrm{Pic}\,(V) \cong H^2(V;\mathbb{Z}) \cong H^4(D;\mathbb{Z}).$$

$\mathrm{Pic}\,(X)$ is isomorphic to the quotient group of $\mathrm{Pic}\,(V)$ modulo the subgroup generated by the classes of irreducible components of D. Hence $\mathrm{Pic}\,(X) = 0$, and X is factorial. Since there are no linear equivalence relations among the irreducible components of D, it follows that $A^* = \mathbb{C}^*$. $\square$

The same argument with the $\mathbb{Z}$-coefficients replaced by $\mathbb{Q}$-coefficients in the proof of Lemma 1.1 shows that *for a $\mathbb{Q}$-homology n-fold X it holds that*

(1) X *is $\mathbb{Q}$-factorial, i.e.,* $\mathrm{Pic}\,(X)$ *is a finite abelian group, and* $A^* = \mathbb{C}^*$.
(2) $H^1(V,\mathcal{O}_V) = H^2(V,\mathcal{O}_V) = 0$.

The following result is an important consequence of a result of Hamm [13] that an affine variety of dimension n defined over $\mathbb{C}$ has the homotopy type of a CW complex of real dimension n.

Lemma 1.2. *Let X be an affine variety of dimension n. Then $H_n(X;\mathbb{Z})$ is a free abelian group of finite rank. Furthermore, $H^i(X;\mathbb{Z}) = 0$ for $i > n$.*

Proof.   The vanishing of $H^i(X;\mathbb{Z})$ follows from the first assertion and the vanishing of $H_i(X;\mathbb{Z})$ for $i > n$. By the universal coefficient theorem, we have

$$H^i(X:\mathbb{Z}) = \mathrm{Hom}(H_i(X;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_{i-1}(X;\mathbb{Z}),\mathbb{Z})\ .$$

If $i = n + 1$, $H^{n+1}(X;\mathbb{Z}) = 0$ because $H_n(X;\mathbb{Z})$ is torsion free. For $i > n + 1$, the result is clear because $H_i(X;\mathbb{Z}) = H_{i-1}(X;\mathbb{Z}) = 0$. $\square$

The following result (see [41, Lemma 1.5] for the proof) is frequently used below.

Lemma 1.3. *Let $f : Y \to X$ be a dominant morphism of algebraic varieties such that the general fibers are irreducible. Then the natural homomorphism $\pi_1(Y) \to \pi_1(X)$ is surjective.*

As an extension of the above argument, one can give a different proof of the following result in [33, Theorem 1]. In the original proof, one has to use, in a crucial step of the proof, a rather difficult result that a smooth, quasi-affine surface X of log Kodaira dimension $-\infty$ is either affine-ruled or contains $\mathbb{A}^2/\Gamma$ as an open set so that $X - \mathbb{A}^2/\Gamma$ is a disjoint union of affine lines, where Γ is a small finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ (cf. the proof of [33]).

Theorem 1.4. *Let the additive group scheme G_a act nontrivially on the affine 3-space $\mathbb{A}^3$. Then $\mathbb{A}^3//G_a \cong \mathbb{A}^2$.*

For the proof, we need two lemmas.

Lemma 1.5. *Let Y be a smooth contractible affine threefold with a nontrivial G_a-action and let $X = Y//G_a$. Let V be a normal projective surface containing X as an open set such that V is smooth along $D := V - X$ and D is a divisor of simple normal crossings. Let $\rho : V' \to V$ be a minimal resolution of singularities of X. Let $p_g(V')$ be the geometric genus of the surface V' which is a birational invariant independent of the choice of V'. Then the following assertions hold.*

    (1) *Both X and X° are simply connected, where X° is the smooth part of X.*

    (2) *$H_1(X; \mathbb{Z}) = 0$. Further, if $p_g(V') = 0$ and $H^1(D; \mathbb{Z}) = 0$, then $H_2(X; \mathbb{Z}) = 0$ and hence X is contractible.*

    (3) *X is smooth under the additional assumptions in the assertion (2).*

Proof. (1) Write $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. Then B is factorial by Lemma 1.1 and hence A is factorial, too. So, the singular locus of X is a finite set of quotient singular points [9, Lemma 3.4]. Hence the quotient morphism $q : Y \to X$ has no fiber components of dimension two and $Y - q^{-1}(X^\circ)$ has dimension ≤ 1. Hence $\pi_1(q^{-1}(X^\circ)) = \pi_1(Y) = (1)$. By Lemma 1.3, both X and X° are simply connected.

(2) Let X' be the inverse image $\rho^{-1}(X)$ and let E be the exceptional locus of ρ which is a divisor with simple normal crossings. Let $F = D + E$. Then we have an exact sequence of cohomology groups

$$H^3(V', F; \mathbb{Z}) \to H^3(V'; \mathbb{Z}) \to H^3(F : \mathbb{Z})$$

where $H^3(F; \mathbb{Z}) = 0$ and $H^3(V', F; \mathbb{Z}) \cong H_1(X^\circ; \mathbb{Z})$ by the Lefschetz duality and $H_1(X^\circ; \mathbb{Z}) = 0$ because $\pi_1(X^\circ) = 1$. Hence $H_1(V'; \mathbb{Z}) \cong H^3(V'; \mathbb{Z}) = 0$. As in the proof of Lemma 1.1, this implies that $H^1(V'; \mathbb{C}) = 0$ and $H^1(V', \mathcal{O}_{V'}) = 0$. Hence we have an exact sequence

$$0 \to H^1(V', \mathcal{O}^*_{V'}) \to H^2(V'; \mathbb{Z}) \to H^2(V', \mathcal{O}_{V'}) \ .$$

Since $H^2(V', \mathcal{O}_{V'}) = 0$ by the hypothesis, we have $\mathrm{Pic}(V') \cong H^1(V', \mathcal{O}_{V'}^*)$ $\cong H^2(V'; \mathbb{Z})$. Namely the group $H^2(V'; \mathbb{Z})$ of topological 2-cocycles is generated by algebraic classes of divisors. Since X is factorial, $\mathrm{Pic}\,(V')$ is generated by the classes of irreducible components of F. Since $H^2(F; \mathbb{Z})$ is a free abelian group generated by the classes of irreducible components of F, the natural homomorphism $H^2(V'; \mathbb{Z}) \to H^2(F; \mathbb{Z})$ is an isomorphism.

Consider the long exact sequence associated to the pair (V', F)

$$H^1(V'; \mathbb{Z}) \to H^1(F; \mathbb{Z}) \to H^2(V', F; \mathbb{Z}) \to H^2(V'; \mathbb{Z}) \to H^2(F; \mathbb{Z}),$$

where $H^1(V'; \mathbb{Z}) = 0$ as $H_1(V'; \mathbb{Z}) = 0$ and $H^2(V'; \mathbb{Z}) \to H^2(F; \mathbb{Z})$ is an isomorphism. Hence $H^1(F; \mathbb{Z}) \cong H_2(V', F; \mathbb{Z})$. Since the divisors D and E are disjoint from each other, $H^1(F; \mathbb{Z}) \cong H^1(D; \mathbb{Z}) \oplus H^1(E; \mathbb{Z})$, where $H^1(D; \mathbb{Z}) = 0$ by the hypothesis and $H^1(E; \mathbb{Z}) = 0$ because X has at worst quotient singular points. So, $H^1(F; \mathbb{Z}) = 0$. By the Lefschetz duality, $H^2(V, F; \mathbb{Z}) \cong H_2(X^\circ; \mathbb{Z})$. It follows that $H_2(X^\circ; \mathbb{Z}) = 0$.

Now let $\{P_1, \ldots, P_r\}$ be the set of singular points of X. Choose a closed neighborhood T_i of P_i for every i so that $T_i \cap T_j = \emptyset$ if $i \neq j$. Let ∂T_i be the boundary of T_i and let $\partial T = \partial T_1 \cup \cdots \cup \partial T_r$. By [40, Proof of Lemma 2.2], we have an exact sequence

$$0 \to H_2(X^\circ; \mathbb{Z}) \to H_2(X; \mathbb{Z}) \to H_1(\partial T; \mathbb{Z}) \to H_1(X^\circ; \mathbb{Z}),$$

where $H_1(X^\circ; \mathbb{Z}) = 0$ as $\pi_1(X^\circ) = 1$ and $H_1(\partial T; \mathbb{Z})$ is a finite abelian group because $\pi_1(\partial T_i)$ is a finite group as the local fundamental group $\pi_{1,P_i}(X)$ of the quotient singular point P_i. Since $H_2(X^\circ; \mathbb{Z}) = 0$ as shown above, $H_2(X; \mathbb{Z})$ is a finite abelian group. Since $H_2(X; \mathbb{Z})$ is torsion free by Lemma 1.2, it follows that $H_2(X; \mathbb{Z}) = 0$.

(3) Since $\pi_1(X) = \pi_1(X^\circ) = 1$ and X is contractible, X is smooth by the so-called *affine Mumford theorem* [8, Theorem 3.6] which we state below. $\square$

Lemma 1.6. *Let X be a normal affine surface such that X is contractible and the smooth part X° of X is simply connected. Then X is smooth.*

Now the different proof of Theorem 1.4 is given as follows. With the notations in Lemma 1.5, let $Y = \mathbb{A}^3$. Choose a general linear hyperplane L in such a way that the quotient morphism q restricted on L is a dominant morphism to $X := \mathbb{A}^3 /\!/ G_a$ and L meets the inverse image $q^{-1}(\mathrm{Sing}X)$ in finitely many points. Then it follows that X is rational and $\overline{\kappa}(X^\circ) = -\infty$. This implies that $p_g(V') = 0$ and $H^1(D; \mathbb{Z}) = 0$ with the notations in Lemma 1.5. Hence X is smooth. It is clear that

A is factorial, $A^* = \mathbb{C}^*$ and $\overline{\kappa}(X) = -\infty$. By a characterization of the affine plane, X is isomorphic to $\mathbb{A}^2$.

REMARK **1.7.** To be accurate, we have to use the result in [33, p. 49] mentioned before Theorem 1.4 in the course of the proof of Lemma 1.6. Hence it is very desirable to give a new proof of Lemma 1.6 which is more topological and not depending on the result used in [33].

In [18, Theorem 2.7], Kaliman and Saveliev state a result stronger than Lemma 1.5. They state that *for every nontrivial G_a-action on a smooth contractible affine algebraic threefold Y, the quotient $X = Y//G_a$ is a smooth contractible affine surface.*

2. $\mathbb{A}^1_*$-FIBRATION

Let $p : Y \to X$ be a dominant morphism of algebraic varieties. We call p an $\mathbb{A}^1$-(resp. $\mathbb{A}^1_*$-) fibration if general fibers of p are isomorphic to $\mathbb{A}^1$ (resp. $\mathbb{A}^1_*$). Here $\mathbb{A}^1_*$ is the affine line with one point deleted. A *singular* fiber of p is a fiber which is not scheme-theoretically isomorphic to $\mathbb{A}^1$ (resp. $\mathbb{A}^1_*$). We say that an $\mathbb{A}^1_*$-fibration is *untwisted* (resp *twisted*) if the generic fiber $Y_K := Y \times_X \operatorname{Spec} K$ has two K-rational places at infinity (resp. if Y_K has one non K-rational place at infinity), where K is the function field of X over $\mathbb{C}$. In the untwisted case, Y_K is isomorphic to $\mathbb{A}^1_{*,K} := \operatorname{Spec} K[t, t^{-1}]$, while Y_K is a non-trivial K-form of $\mathbb{A}^1_{*,K}$ and $\operatorname{Pic}(Y_K) \cong \mathbb{Z}/2\mathbb{Z}$ in the twisted case. If Y is a factorial affine variety, then any $\mathbb{A}^1_*$-fibration on Y is untwisted because Y_K is then factorial. In fact, if $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ then B is factorial and hence the quotient ring $B \otimes_A K$ is factorial. Since $Y_K = \operatorname{Spec} B \otimes_A K$, Y_K is factorial. In the present article, an $\mathbb{A}^1_*$-fibration is *always assumed to be untwisted.*

If the additive group scheme G_a (resp. the multiplicative group scheme G_m) acts on an affine threefold Y then the algebraic quotient $Y//G_a$ (resp. $Y//G_m$) exists. Namely, if $Y = \operatorname{Spec} B$ and X is the quotient of Y by G_a (resp. G_m), then the invariant subring $A = B^{G_a}$ (resp. $A = B^{G_m}$) is the coordinate ring of X and the quotient morphism $q : Y \to X$ is given by the inclusion $A \hookrightarrow B$. Note that A is finitely generated over $\mathbb{C}$ by a lemma of Zariski in the case of G_a [49] and by the well-known result on reductive group actions in the case of G_m. In the case of G_m-actions, we assume *unless otherwise mentioned* that $\dim A = \dim B - 1$. Hence q has relative dimension one. In the case of a G_a-action, the quotient morphism $q : Y \to X$ is an $\mathbb{A}^1$-fibration which is not necessarily surjective, but in the case of a G_m-action, q is surjective and either an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration according as general orbits admit fixed points or not. If the quotient

morphism $q : Y \to X$ is an $\mathbb{A}^1_*$-fibration, it is untwisted by a theorem of Rosenlicht [43]. We recall the following fundamental result on the quotients under reductive algebraic groups [2]. When we consider the quotient morphism by G_a (or G_m), we denote it by q in most cases, but an $\mathbb{A}^1$-fibration (or $\mathbb{A}^1_*$-fibration) by p.

Lemma 2.1. *Let G be a connected reductive algebraic group acting on a smooth affine algebraic variety Y and let $q : Y \to X$ be the quotient morphism. Then, for two points P_1, P_2, we have $q(P_1) = q(P_2)$ if and only if $\overline{GP_1} \cap \overline{GP_2} \neq \emptyset$. Hence for any point Q of X, the fiber $q^{-1}(Q)$ contains a unique closed orbit.*

In order to distinguish the case of surfaces from the case of threefolds, we use the notation like $p : X \to C$ in the case of surfaces and retain the notation $p : Y \to X$ for the case of threefolds or varieties in general.

Lemma 2.2. *Let X be a normal affine surface and let $p : X \to C$ be an $\mathbb{A}^1_*$-fibration. Then the following assertions hold.*
  (1) *A singular fiber F of p is written in the form $F = \Gamma + \Delta$, where $\Gamma = m\mathbb{A}^1_*, \emptyset$ or $m\mathbb{A}^1 + n\mathbb{A}^1$ with two $\mathbb{A}^1$'s meeting in one point which may be a cyclic quotient singular point of X, $\Gamma \cap \Delta = \emptyset$ and Δ is a disjoint union of affine lines with multiplicities, each of which may have a unique cyclic singular quotient point of X. Each type of a singular fiber is realizable.*
  (2) *If p is the quotient morphism of a G_m-action on X and p is an $\mathbb{A}^1_*$-fibration, the part Δ is absent in a singular fiber F. Hence F is either $m\mathbb{A}^1_*$ or $m\mathbb{A}^1 + n\mathbb{A}^1$ in the above list.*

Proof.   (1) In [36, Lemma 4] and [48, Proposition 5.1 (b)], the case where X is a smooth affine surface is treated. In [40, Lemma 2.9], singular fibers of an $\mathbb{A}^1_*$-fibration are classified, where singularities of the surface X are assumed to be quotient singularities, but one can drop this condition and make the same argument by assuming simply that X is normal. The normal case is also treated in [4, Proposition 3.8 and Theorem 4.18].

(2) By the assertion (1) and Lemma 2.1, the only possible cases of a singular fiber of the quotient morphism p are the two cases listed in the statement and the case where F_{red} is irreducible and isomorphic to $\mathbb{A}^1$ which may contain a unique cyclic quotient singularity. We show that the last case does not occur. Suppose that $F_{\mathrm{red}} \cong \mathbb{A}^1$. Then F contains a unique fixed point P. Suppose first that X is smooth at P. Look at the induced tangential representation of G_m on $T_{X,P}$ which is diagonalizable and has weights $-a, b$ with $a, b > 0$. Then it follows that F is locally near P a union of two irreducible

components meeting in P. This is a contradiction. Suppose that X is singular at P. Let $\sigma : \widetilde{X} \to X$ be a minimal resolution of singularity at P. Then the inverse image $\sigma^{-1}(F_{\mathrm{red}})$ is the proper transform G of F_{red} and a linear chain of $\mathbb{P}^1$'s with G meeting one of the terminal components of the linear chain. Since the G_m-ation on X lifts to $\widetilde{X}$, each irreducible component of $\sigma^{-1}(F_{\mathrm{red}})$ is G_m-stable and the other terminal component, say $\widetilde{F}$, of the linear chain has an isolated fixed point, say $\widetilde{P}$. Note that we obtain an affine surface from $\widetilde{X}$ by removing all components of $\sigma^{-1}(F_{\mathrm{red}})$ except for the component $\widetilde{F}$. Now looking at the induced tangential representation of G_m on $T_{\widetilde{X},\widetilde{P}}$, we have a contradiction as in the above smooth case. So, the case $F_{\mathrm{red}} \cong \mathbb{A}^1$ does not occur. $\qquad\square$

Special attention has to be paid in the case where the G_m-quotient morphism $q : Y \to X$ is an $\mathbb{A}^1$-fibration. We consider the surface case first and then treat the case of threefolds.

Lemma 2.3. *Let X be a normal affine surface with a G_m-action. Suppose that the quotient morphism $\rho : X \to C$ is an $\mathbb{A}^1$-fibration, where C is a normal affine curve. Then the following assertions hold.*

(1) *There exists a closed curve Γ on X such that the restriction of q onto Γ induces an isomorphism between Γ and C, i.e., Γ is a cross-section of ρ.*

(2) *Suppose further that X is smooth. Then every fiber of q is reduced and isomorphic to $\mathbb{A}^1$. Hence X is a line bundle over C.*

(3) *Suppose that X is smooth and C is rational. Then X is isomorphic to a direct product $C \times \mathbb{A}^1$.*

Proof. (1) Let $\overline{\rho} : \overline{X} \to \overline{C}$ be a G_m-equivariant completion of $\rho : X \to C$. Namely, $\overline{X}$ is a normal projective surface containing X as an open set, $\overline{C}$ is the smooth completion of C. We may assume that $\overline{X}$ is smooth along $\overline{X} \setminus X$, ρ extends to a morphism $\overline{\rho}$ and the G_m-action extends to $\overline{X}$ so that $\overline{\rho}$ is a $\mathbb{P}^1$-fibration. Since ρ is an $\mathbb{A}^1$-fibration, there exists an open set U of C such that $\rho^{-1}(U) \cong U \times \mathbb{A}^1$ with G_m acting on $\mathbb{A}^1$ in a natural fashion. Let Γ_0 be the fixed point locus in $\rho^{-1}(U)$ and let Γ (resp. $\overline{\Gamma}$) be the closure of Γ_0 in X (resp. $\overline{X}$). Then the restriction of $\overline{\rho}$ onto $\overline{\Gamma}$ is a birational morphism onto $\overline{C}$, hence an isomorphism. Thus $\overline{\Gamma}$ is a cross-section of $\overline{\rho}$. On the other hand, $\overline{\rho}$ has another cross-section $\overline{\Gamma}_\infty$ lying outside X. Since ρ is an $\mathbb{A}^1$-fibration, every fiber of ρ is a disjoint union of the affine lines, hence it consists of a single affine line by Lemma 2.1. Suppose that for a point $\alpha \in C$, the fiber F_α does not meet Γ. Then $\overline{\Gamma}$ meets the fiber $\overline{F}_\alpha := \overline{\rho}^{-1}(\alpha)$

outside F_α, and F_α has an isolated G_m-fixed point P. This leads to a contradiction if we argue as in the proof of Lemma 2.2. Namely, if X is smooth at P, then consider the induced tangential representation at P. Otherwise, consider a minimal resolution of singularity at P and a G_m-fixed point appearing in one of the terminal components of the exceptional locus which does not meet the proper transform of F_α. Thus Γ meets every fiber of ρ and is smooth.

(2) With the above notations, suppose that X has a singular point on the fiber F_α. Since G_m acts transitively on $F_\alpha - F_\alpha \cap \Gamma$, X has cyclic singularity at the point $P_\alpha := F_\alpha \cap \Gamma$. Let $\sigma : \widetilde{X} \to \overline{X}$ be the minimal resolution of singularities and let $\Delta = \sigma^{-1}(P_\alpha)$. The $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{X} \to \overline{C}$ lifts to a $\mathbb{P}^1$-fibration $\widetilde{\rho} : \widetilde{X} \to \overline{C}$. The proper transform $\widetilde{\Gamma}$ of $\overline{\Gamma}$ is a cross-section of $\widetilde{\rho}$. The fiber $\widetilde{F}_\alpha$ of $\widetilde{\rho}$ corresponding to $\overline{F}_\alpha$ contains the linear chain Δ in such a way that one terminal component, say G, meets $\widetilde{\Gamma}$ and the other terminal component meets the proper transform of F_α. Further, all the components except for the terminal component G meeting $\widetilde{\Gamma}$ can be contracted smoothly. In fact, we can replace the fiber F_α by $G - \{Q\}$ without losing the affineness of X, where Q is the point where G meets the adjacent component of $\widetilde{F}_\alpha$. If X is smooth along F_α, it is reduced. If X is smooth, the $\mathbb{A}^1$-fibration ρ has no singular fibers. Hence X is an $\mathbb{A}^1$-bundle over C [9, Lemma 1.15]. Since the two cross-sections $\overline{\Gamma}$ and $\overline{\Gamma}_\infty$ are disjoint from each other over C, it follows that X is a line bundle over C.

(3) Since Pic $(C) = 0$, every line bundle is trivial. $\qquad\square$

The proof of the assertion (2) implies that every normal affine surface with a G_m-action and an $\mathbb{A}^1$-fibration as the quotient morphism is constructed from a line bundle over C by a succession of blowing-ups with centers on the zero section and contractions of linear chains. The following result is well-known.

Lemma 2.4. *Let* $Y = \mathrm{Spec}\, B$ *be a smooth affine threefold with a* G_m-*action and let* $q : Y \to X := \mathrm{Spec}\, A$ *be the quotient morphism. Assume that* q *is an* $\mathbb{A}^1$-*fibration and that* q *is equi-dimensional. Then the following assertions hold.*

(1) *We may assume that* B *is positively graded, that is,* B *is a graded* A-*algebra indexed by* $\mathbb{Z}_{\geq 0}$.
(2) A *is normal and factorially closed in* B.
(3) $q : Y \to X$ *has a cross-section* S. *Namely, each fiber of* q *has dimension one and meets* S *in one point transversally.*
(4) X *is smooth, and* Y *is a line bundle over* X.

(5) *If Y is factorial, then the equi-dimensionality condition is automatically satisfied and Y is isomorphic to a direct product $X \times \mathbb{A}^1$ with G_m acting on the factor $\mathbb{A}^1$ in a natural fashion.*

Proof. (1) By [9, Lemma 1.2], there exists a nonzero element $a \in A$ such that $B[a^{-1}] = A[a^{-1}][u]$, where u is an element of B algebraically independent over the quotient field $Q(A)$. In fact, u is a variable on a general fiber of q. We may assume that the G_m-action on the fiber is given by ${}^t u = tu$. Note that the G_m-action gives a $\mathbb{Z}$-graded ring structure on B. Let b be a homogeneous element of B. We can write $a^r b = f(u)$, where $f(u) = a_0 u^m + a_1 u^{m-1} + \cdots + a_m \in A[u]$ with $a_0 \neq 0$. Then we have

$$a^r ({}^t b) = f(tu) = a_0 t^m u^m + a_1 t^{m-1} u^{m-1} + \cdots + a_m .$$

Hence $a^r b = a_0 u^m$ and b has degree m. This implies that every homogeneous element of B has degree ≥ 0.

(2) The normality of A is well-known [27, p.100], and the factorial closedness follows from the assertion (1).

(3) Let S_0 be the closed set in the open set $q^{-1}(D(a))$ defined by $u = 0$. Hence S_0 is the locus of the fixed points in the fibers $q^{-1}(Q)$ when $Q \in D(a)$. Let S be the closure of S_0 in Y. Then S is a rational cross-section of q. We shall show that S meets every fiber of q in one point. Let P be a point of X, let C be a general irreducible curve on X passing through P and let $Z = Y \times_X C$. Then $q_C : Z \to C$, the projection onto C, is an $\mathbb{A}^1$-fibration. Let $\nu_Z : \widetilde{Z} \to Z$ and $\nu_C : \widetilde{C} \to C$ be the normalizations of Z and C respectively. Then there exists an $\mathbb{A}^1$-fibration $\rho : \widetilde{Z} \to \widetilde{C}$ such that $\nu_C \cdot \rho = q_C \cdot \nu_Z$. The normal affine surface $\widetilde{Z}$ has the induced G_m-action and ρ is the quotient morphism. Choosing the open set $D(a)$ small enough, we may assume that $C \cap D(a)$ is a non-empty set contained in the smooth part of C. Let $\Gamma_0 = S_0 \cap Z$ and let $\widetilde{\Gamma}$ be the closure of Γ_0 in $\widetilde{Z}$. Let $\widetilde{P}$ be a point of $\widetilde{C}$ lying over P. Let $\widetilde{F} = \rho^{-1}(\widetilde{P})$ and $F = q^{-1}(P)$. By Lemma 2.3, $\widetilde{\Gamma}$ meets $\widetilde{F}$ in a single point. This implies that S meets F in a point, say R. By [9, Theorem 1.15 and Lemma 3.5] and Lemma 2.1, F consists of one irreducible component which is contractible and smooth outside the point R because it admits a non-trivial G_m-action with a fixed point R. Since Y is smooth, the local intersection multiplicity $i(S, F; R) = 1$. This implies that F is reduced and smooth also at R. Hence F is isomorphic to $\mathbb{A}^1$.

(4) The morphism $q : Y \to X$ induces a quasi-finite birational morphism $q|_S : S \to X$. Then it is an isomorphism by Zariski's main

theorem. Since S is smooth by the argument in the proof of the asser-
tion (3), X is also smooth. Furthermore, every fiber of q is an affine
line. Hence Y is an $\mathbb{A}^1$-bundle over X. Since a smooth completion
$\overline{q} : \overline{Y} \to \overline{X}$ is a $\mathbb{P}^1$-fibration with two cross-sections (the zero section
and the infinity section) which are disjoint over X, Y is a line bundle
over X.

(5) By [9, Lemma 1.10], q does not have codimension one fiber com-
ponents. Since X is factorial, Y is isomorphic to $X \times \mathbb{A}^1$. This fact
follows from the proof of the assertion (1). In fact, one can take the
element u to be a prime element of B. Then, the relation $a^r b = a_0 u^m$
for a homogeneous element b of degree m implies that $b = cu^m$ for a
certain element $c \in A$. Hence $B = A[u]$. $\square$

In [9, Theorem 1.4], it is shown that any $\mathbb{A}^1$-fibration $p : Y \to X$ on
an affine threefold Y is factored by the quotient morphism $q : Y \to Z$
by a certain G_a-action on Y. Meanwhile, this is not the case with an
$\mathbb{A}^1_*$-fibration. Examples can be easily produced by using Lemma 2.2,
(1).

A homology threefold Y is a *contractible threefold* if it is topologi-
cally contractible. A homology threefold is contractible if and only if
$\pi_1(Y) = 1$. We consider a G_m-action on such a threefold.

Lemma 2.5. *Let G_m act nontrivially on a $\mathbb{Q}$-homology threefold Y.
Then the fixed point locus Y^{G_m} is a non-empty connected closed sub-
set of X. Furthermore, $H_i(Y^{G_m}; \mathbb{Z}/p\mathbb{Z}) = 0$ for every $i > 0$ and for
infinitely many prime numbers p.*

Proof.   Since $H_i(Y; \mathbb{Z})$ is a finite group, there exist infinitely many
primes p such that Y is $\mathbb{Z}/p\mathbb{Z}$-acyclic, i.e., $H_i(Y; \mathbb{Z}/p\mathbb{Z}) = 0$ for every
$i > 0$. We choose such a prime p with the additional property that
p does not divide any weight of the induced tangential G_m-action on
$T_{Y,Q}$ for every fixed point Q. We take a point $Q \in Y^{G_m}$. By [24,
Theorem 1], there exists a G_m-stable open neighborhood U of Q in the
Euclidean topology and algebraic functions x, y, z around the point Q
which form a system of analytic coordinates on U. The G_m-action is
given by $^t(x, y, z) = (t^a x, t^b y, t^c z)$ with integers a, b, c, where $p \nmid \alpha$ for
any $\alpha \in \{a, b, c\} \setminus \{0\}$. Then $U \cap Y^{G_m} = U \cap Y^{\mathbb{Z}/p\mathbb{Z}}$. Since $Y^{G_m} \subseteq Y^{\mathbb{Z}/p\mathbb{Z}}$,
it follows that Y^{G_m} is a connected component of $Y^{\mathbb{Z}/p\mathbb{Z}}$. By [5], we have
the inequality

$$\sum_i \dim H_i(Y^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) \leq \sum_i \dim H_i(Y; \mathbb{Z}/p\mathbb{Z}) = 1.$$

This implies that $Y^{\mathbb{Z}/p\mathbb{Z}}$ is connected and equal to Y^{G_m}. Hence Y^{G_m} is
connected and $H_i(Y^{G_m}; \mathbb{Z}/p\mathbb{Z}) = 0$ for every $i > 0$. $\square$

We have the following result.

Theorem 2.6. *Let G_m act nontrivially on a $\mathbb{Q}$-homology threefold Y and let $q : Y \to X$ be the quotient morphism. Suppose that q has relative dimension one. Then the following assertions hold.*

(1) *Y^{G_m} is $\mathbb{Q}$-acyclic. If Y is a homology threefold, then Y^{G_m} is $\mathbb{Z}$-acyclic.*

(2) *If $\dim Y^{G_m} = 2$ then $Y^{G_m} \cong X$ and Y is a line bundle over X. If Y is a homology threefold, then $Y \cong X \times \mathbb{A}^1$.*

(3) *If $\dim Y^{G_m} = 1$ then $Y^{G_m} \cong \mathbb{A}^1$ and $q|_{Y^{G_m}} : Y^{G_m} \to X$ is a closed embedding. If Y is contractible and q is equi-dimensional, then X is a smooth contractible surface of log Kodaira dimension $-\infty$ or 1.*

(4) *If $\dim Y^{G_m} = 0$ then the G_m-action on Y is hyperbolic, i.e., the tangential representation on $T_{Y,Q}$ for the unique fixed point Q has mixed weights, e.g., either $a_1 < 0, a_2 > 0$ and $a_3 > 0$, or $a_1 > 0, a_2 < 0$ and $a_3 < 0$. In this case, there is an irreducible component of codimension one contained in the fiber of q passing through the point Q.*

Proof. (1) The $\mathbb{Q}$-acyclicity is verified below case by case according to $\dim Y^{G_m}$. When X is a homology threefold, the acyclicity of Y^{G_m} follows from the Smith theory [44, §22.3].

(2) By Lemma 2.1, Y^{G_m} lies horizontally to the morphism q and the restriction $q|_{Y^{G_m}} : Y^{G_m} \to X$ is a bijection (hence a birational morphism). In particular, Y^{G_m} is irreducible. Further, $q : Y \to X$ is an $\mathbb{A}^1$-fibration. Since Y is $\mathbb{Q}$-factorial by a remark after Lemma 1.1, q is equi-dimensional (cf. [9, Lemma 1.14]). Then Lemma 2.4 implies that X is smooth and $q : Y \to X$ is a line bundle. By Zariski's main theorem, we conclude that $Y^{G_m} \cong X$. Furthermore, since Y is contractible to X, it follows that Y^{G_m} is $\mathbb{Q}$-acyclic (resp. $\mathbb{Z}$-acyclic) if Y is a $\mathbb{Q}$-homology threefold (resp. $\mathbb{Z}$-homology threefold). If Y is a homology threefold, then X is factorial and a line bundle over X is trivial. Hence $Y \cong X \times \mathbb{A}^1$.

(3) Y^{G_m} is a connected curve. Further, Y^{G_m} is smooth. In fact, the smoothness of the fixed point locus is a well-known fact for a reductive algebraic group action on a smooth affine variety (cf. [44])[1]. Hence Y^{G_m} is irreducible. Then Y^{G_m} is an affine line because $H_1(Y^{G_m}; \mathbb{Z}/p\mathbb{Z}) = 0$.

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[1]In the case where G is a connected reductive algebraic group acting on a smooth affine variety Y, the G-action near a fixed point Q is locally analytically G-equivalent to a linear representation [24, 30]. In the case of a linear representation of G on the affine space $\mathbb{C}^n$, the fixed point locus near the origin is a linear subspace and it is G-equivariantly a direct summand of $\mathbb{C}^n$ by the complete reducibility of

By Lemma 2.1, q induces a closed embedding of Y^{G_m} into X [2]. If Y is contractible, X is contractible by [28, Theorem B]. By Lemma 1.3, $\pi_1(X^\circ) = (1)$, where X° is the smooth part of X. Then X is smooth by Lemma 1.6. So, X is a smooth contractible surface containing a curve isomorphic to $\mathbb{A}^1$. Hence X has log Kodaira dimension $-\infty$ or 1 (cf. [48, 10]).

(4) The induced G_m-action on the tangent space $T_{Y,Q}$ of the unique fixed point Q must have mixed weights. Otherwise, $\dim Y^{G_m} > 0$ near the point Q. This is a contradiction. To prove the last assertion, we have only to show it when Y is idetified with the tangent linear 3-space $T_{Y,Q}$. Then the G_m-action is diagonalized and $^t(x, y, z) = (t^{a_1}x, t^{a_2}y, t^{a_3}z)$ with respect to a suitable system of coordinates (x, y, z). Then the invariant elements, viewed as elements in $\Gamma(Y, \mathcal{O}_Y)$, are divisible by x. Hence the fiber F of q passing through Q contains an irreducible component $\{x = 0\}$ which is a hypersurface of Y. □

Question 2.7. In the case (3) above, is Y isomorphic G_m-equivariantly to a direct product $Z \times Y^{G_m}$, where Z is a smooth affine surface with G_m acting on it? If this is the case, q is the direct product $q = q' \times Y^{G_m}$ of the quotient morphism $q' : Z \to C$, where $C = Z//G_m$.

We discuss this question in Section 4 in detail. Theorem 2.6 has the following consequence.

Corollary 2.8. *Let the notations and the assumptions be the same as in Theorem 2.6. If the quotient morphism $q : Y \to X$ has equidimension one, then the fixed point locus Y^{G_m} has positive dimension.*

Now we consider an $\mathbb{A}^1_*$-fibration $p : Y \to X$, where Y is a smooth affine threefold and X is a normal affine surface. For a point P, the scheme-theoretic fiber $Y \times_X \operatorname{Spec} k(P)$ is denoted by $p^{-1}(P)$ or F_P, where $k(P)$ is the residue field of P in X. A fiber F_P of p is called

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G. The smoothness of the fixed point locus Y^G near the point Q follows from this observation.

[2]Let $Q \in Y^{G_m}$. In view of [24], there exists a system of local (analytic) coordinates $\{x, y, z\}$ at Q such that G_m acts as $^t(x, y, z) = (x, t^{-a}y, t^b z)$ with $ab > 0$ and $y = z = 0$ defining the curve Y^{G_m} near Q. Then the quoteint surface X at the point $P = q(Q)$ has a system of local analytic coordinates $\{x, x^{b'}y^{a'}\}$ with $a' = a/d, b' = b/d$ and $d = \gcd(a, b)$. Then the curve $q(Y^{G_m})$ is defined by $x^{a'}y^{b'} = 0$ near P and X is smooth near the curve $q(Y^{G_m})$. Hence Y^{G_m} and $q(Y^{G_m})$ are locally isomorphic at Q and P. Since $q|_{Y^{G_m}} : Y^{G_m} \to X$ is injective, it is a closed embedding. Furthermore, the fiber of q through the point Q is a cross $a'\mathbb{A}^1 + b'\mathbb{A}^1$. See the definition after Corollary 2.8.

singular if F_P is not isomorphic to $\mathbb{A}^1_*$ over $k(P)$. The set of points $P \in X$ such that F_P is singular is denoted by $\mathrm{Sing}^{st}(p)$ and called the *strict singular locus* or the *degeneracy locus* of p. For a technical reason (cf. Lemma 2.13 and a remark below it), we define the *singular locus* $\mathrm{Sing}(p)$ as the union of $\mathrm{Sing}^{st}(p)$ and the set $\mathrm{Sing}(X)$ of singular points. A singular fiber F_P is called a *cross* (resp. *tube*) [3] if it has the form $F_P \cong m\mathbb{A}^1 + n\mathbb{A}^1$ (resp. $m\mathbb{A}^1_*$). When we speak of a cross on a threefold or a surface which is smooth at the intersection point Q of two lines, we assume that two affine lines meet each other transversally at Q. By abuse of the terminology, we call a singular fiber F on a normal surface a cross even when the surface has a cyclic quotient singularity at the intersection point Q and the proper transforms of the two lines on the minimal resolution of singularity form a linear chain together with the exceptional locus.

We first recall a result of Bhatwadekar-Dutta [1, Theorem 3.11].

Lemma 2.9. *Let R be a Noetherian normal domain and let A be a finitely generated flat R-algebra such that*

(1) *The generic fiber $K \otimes_R A$ is a Laurent polynomial ring $K[T, T^{-1}]$ in one variable over $K = Q(R)$.*
(2) *For each prime ideal $\mathfrak{p}$ of R of height one, the fiber $\mathrm{Spec}\, A \otimes_R k(\mathfrak{p})$ is geometrically integral but is not $\mathbb{A}^1$ over $k(\mathfrak{p})$.*

Then there exists an invertible ideal I in R such that A is a $\mathbb{Z}$-graded R-algebra isomorphic to the R-subalgebra $R[IT, I^{-1}T^{-1}]$ of $K[T, T^{-1}]$. In particular, $\mathrm{Spec}\, A$ is locally $\mathbb{A}^1_*$ and hence an $\mathbb{A}^1_*$-fibration over $\mathrm{Spec}\, R$.

In geometric terms, a weaker version of Lemma 2.9 in the case of dimension three is stated as follows.

Lemma 2.10. *Let $p : Y \to X$ be an $\mathbb{A}^1_*$-fibration with a smooth affine threefold Y and let P be a closed point of X. Suppose that the following conditions are satisfied.*

(1) *There is an open neighborhood U of P such that U is smooth and p is equi-dimensional over U.*
(2) *Every fiber $F_{P'}$ of p for $P' \in U \setminus \{P\}$ is isomorphic to $\mathbb{A}^1_*$.*

Then the fiber F_P is isomorphic to $\mathbb{A}^1_*$ and $p : p^{-1}(U) \to U$ is an $\mathbb{A}^1_*$-bundle.

Proof. By replacing X by a smaller affine open neighborhood of P contained in U, we may assume that X is smooth, p is equi-dimensional

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[3]The naming of cross is apparent. The fiber $m\mathbb{A}^1_*$ is a projective line with two end points lacking and thickened with multiplicity. It looks like a tube.

and $\mathrm{Sing}(p) \cap (X \setminus \{P\}) = \emptyset$. Then p is flat. Further, there exists a $\mathbb{P}^1$-fibration $\overline{p} : \overline{Y} \to X$ such that Y is an open set of $\overline{Y}$ and $\overline{p}|_Y = p$. Since the $\mathbb{A}^1_*$-fibration p is assumed to be untwisted, the generic fiber $\overline{Y}_K$ has two K-rational points ξ_1, ξ_2, where K is the function field of X over $\mathbb{C}$. Let S_i be the closure of ξ_i in $\overline{Y}$ for $i = 1, 2$. By the assumption (2) above, S_1, S_2 are cross-sections over $X \setminus \{P\}$. Namely, $p^{-1}(X \setminus \{P\})$ is an $\mathbb{A}^1_*$-bundle. Hence $p^{-1}(X \setminus \{P\}) \cong \mathcal{S}pec(\mathcal{O}_{X \setminus \{P\}}[\mathcal{L}', \mathcal{L}'^{-1}])$, where $\mathcal{L}'$ is an invertible sheaf on $X \setminus \{P\}$. Since X is smooth and P is a point of codimension two, $\mathcal{L}'$ is extended to an invertible sheaf $\mathcal{L}$ on X. Since Y is affine and codim $_Y p^{-1}(P) = 2$, it follows that $Y \cong \mathcal{S}pec(\mathcal{O}_X[\mathcal{L}, \mathcal{L}^{-1}])$. So, $p : Y \to X$ is an $\mathbb{A}^1_*$-bundle and $F_P \cong \mathbb{A}^1_*$. $\qquad\qquad \square$

Let F_P be a fiber of an $\mathbb{A}^1_*$-fibration $p : Y \to X$ with $P \in X$. Let C be a general smooth curve on X passing through P. Let Z be the normalization of $Y \times_X C$. Then p induces an $\mathbb{A}^1_*$-fibration $p_C : Z \to C$ and the fiber $\widetilde{F}_P$ over the point P is a finite covering of the fiber F_P (in the sense that the normalization morphism induces a finite morphism $\widetilde{F}_P \to F_P$. In view of Lemma 2.2, we say that the $\mathbb{A}^1_*$-fibration p is *unmixed* if $\widetilde{F}_P$ for every $P \in X$ is $\mathbb{A}^1_*$, $\mathbb{A}^1 + \mathbb{A}^1$, or a disjoint union of $\mathbb{A}^1$'s when taken with reduced structure. This definiton does not depend on the choice of the curve C.

Lemma 2.11. *Let $p : Y \to X$ be an unmixed $\mathbb{A}^1_*$-fibration such that Y and X are smooth. Then the singular locus $\mathrm{Sing}(p)$ is a closed subset of X of codimension one.*

Proof.   Lemma 2.9 or Lemma 2.10 implies that every irreducible component of the closure $\overline{\mathrm{Sing}(p)}$ of $\mathrm{Sing}(p)$ has codimension one.

So, we have only to show that $\mathrm{Sing}(p)$ is a closed set. It suffices to show that if the fiber F_P over $P \in X$ is $\mathbb{A}^1_*$ then there exists an open neighborhood U of P such that $F_{P'} \cong \mathbb{A}^1_*$ for every $P' \in U$. Let C be a curve passing through P. If C is not a component of $\overline{\mathrm{Sing}(p)}$, then the fiber of p over the generic point of C is $\mathbb{A}^1_*$ and hence geometrically integral. Suppose that C is a component of $\overline{\mathrm{Sing}(p)}$. The unmixedness condition on p implies that for a general point P' of C, the fiber $F_{P'}$ is either irreducible and dominated by $\mathbb{A}^1_*$ (case (i)), or each irreducible component of $F_{P'}$ is dominated by a contractible curve (case (ii)). Suppose that the case (ii) occurs. Then there exists a normal affine surface $\widetilde{Z}$ with an $\mathbb{A}^1$-fibration $\widetilde{p} : \widetilde{Z} \to \widetilde{C}$ satisfying the

following commutative diagram

$$
\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{\;\mu\;} & Y \times_X C \\
\Big\downarrow{\scriptstyle \widetilde{p}} & & \Big\downarrow{\scriptstyle p_C} \\
\widetilde{C} & \xrightarrow{\;\nu\;} & C
\end{array}
$$

where μ is a quasi-finite morphism and ν is the normalization morphism. Since any singular fiber of an $\mathbb{A}^1$-fibration on a normal affine surface is a disjoint union of the affine lines, it turns out that the fiber F_P, which is a fiber of p_C in the above diagram, is dominated by the affine line, which is a connected component of the fiber of $\widetilde{p}$. This is a contradiction, and the case (ii) cannot occur. Now removing from X all irreducible components of $\overline{\mathrm{Sing}(p)}$ for whose general points the case (ii) occurs and replacing Y by the inverse image of the open set of X thus obtained, we may assume that the case (i) occurs for general points of the irreducible components of $\overline{\mathrm{Sing}(p)}$. Let

$$dp : \mathcal{T}_Y \to p^* \mathcal{T}_X$$

be the tangential homomorphism of the tangent bundles on Y and X. Let $\mathcal{C}$ be the cokernel of dp. Then $\mathcal{C}$ is a coherent $\mathcal{O}_Y$-Module. Hence $\mathrm{Supp}\,(\mathcal{C})$ is a closed set T such that $T = p^{-1}(p(T))$ and $T = \overline{\mathrm{Sing}(p)}$. Hence the point P belongs to $p(T)$. However, dp is everywhere surjective on the fiber F_P and $F_P \not\subset \mathrm{Supp}\,(\mathcal{C})$. This is a contradiction.

By the above argument, every irreducible curve C through the point P has the property that the general fibers are isomorphic to $\mathbb{A}^1_*$. Hence the fiber of p over the generic point of C is geometrically integral. By Lemma 2.9, we know that $B \otimes_A \mathcal{O}_P \cong \mathcal{O}_P[x, x^{-1}]$. This implies that there exists an open neighborhood U of P satisfying the required property. $\qquad\square$

In the proofs of Lemmas 2.9, 2.10 and 2.11, the flatness condition on the morphism p should not be overlooked. Hence the assumption that X be normal instead of being smooth does not seem to be sufficient for the conclusion. So we ask the following question.

Question 2.12. Let $p : Y \to X$ be an $\mathbb{A}^1_*$-fibration with a smooth affine threefold Y. Is a point $P \in X$ smooth if the fiber F_P is isomorphic to $\mathbb{A}^1_*$?

The following is a partial answer to this question.

Lemma 2.13. *Let Y be a smooth affine threefold with an effective G_m-action and let $q : Y \to X$ be the quotient morphism. Suppose that the G_m-action is fixed point free. Then X has at worst cyclic quotient singular points.*

Proof.   Suppose that P is a singular point of X. Let Q be a point of Y such that $q(Q) = P$ and let G_Q be the isotropy group at Q which is a finite cyclic group. Then, by Luna's étale slice theorem [30], there exists an affine subvariety V of Y with a G_Q-action and an étale morphism $\varphi/G_Q : V/G_Q \to X$ such that $Q \in V$, V is smooth at Q and $(\varphi/G_Q)(Q) = P$. This implies that X has at worst cyclic quotient singularity at P. Since the ground field is $\mathbb{C}$, we can work with an analytic slice instead of an étale slice. $\square$

We can easily show that if X is smooth at P then the fiber F_P has multiplicity equal to $m = |G_Q|$ near the point Q. In fact, if V is an analytic slice with coordinates x, y, the G_Q-action on V is given by $\zeta(x, y) = (\zeta x, \zeta^b y)$ with a generator ζ of G_Q which is identified with an m-th primitive root of unity, where $0 \le b < m$. Then P is singular if and only if $b > 0$. Hence $b = 0$ if P is smooth. This implies that (x^m, y) is a local system of parameters of X at P. Hence the fiber F_P has multiplicity m. If X is singular at P, the fiber F_P is a multiple fiber. In fact, with the above notations, $\mathfrak{m}_{X,P}\widehat{\mathcal{O}}_{Y,Q}$ is generated by

$$\{x^m, y^m\} \cup \{x^s y^n \mid bn + s \equiv 0 \pmod{m}, \ 0 < s < m, \ 0 < n < m\}.$$

Since $\widehat{\mathcal{O}}_{Y,Q} = \mathbb{C}[[x, y, z]]$ with a fiber coordinate z of F_P, it follows that $\mathfrak{m}_{X,P}\widehat{\mathcal{O}}_{Y,Q}$ is not a radical ideal. Hence the fiber F_P is not reduced near the point Q.

The singular locus $\mathrm{Sing}(p)$ may consist of a single point as shown in the following example.

EXAMPLE **2.14.** Let $\mathbb{A}^3 \to \mathbb{A}^2$ be the morphism defined by $(x, y, z) \mapsto (xy, x^2(xz + 1))$, where (x, y, z) is a system of coordinates of $\mathbb{A}^3$. Then p is an $\mathbb{A}^1_*$-fibration and $\mathrm{Sing}(p) = \{(0, 0)\}$. In fact, set $\alpha = xy$, $\beta = x^2(xz+1)$. Then $p^{-1}(\alpha, \beta) = \{(\alpha y^{-1}, y, \alpha^{-3}y(\beta y^2 - \alpha^2)) \mid y \in \mathbb{C}^*\} \cong \mathbb{A}^1_*$ if $\alpha \ne 0$, $p^{-1}(0, \beta) = \{y = 0, \ x^2(xz+1) = \beta\} \cong \mathbb{A}^1_*$ if $\alpha = 0$, $\beta \ne 0$ and $p^{-1}(0, 0) = \{x = 0\} \cup \{y = xz + 1 = 0\} \cong \mathbb{A}^2 \cup \mathbb{A}^1_*$ if $\alpha = \beta = 0$. $\square$

The following example shows that a disjoint union of affine lines may appear as a singular fiber of an $\mathbb{A}^1_*$-fibration.

EXAMPLE **2.15.** Let $p : \mathbb{A}^3 \to \mathbb{A}^2$ be the morphism defined by $(x, y, z) \mapsto (xy, x^2(yz + 1))$. Then p is an $\mathbb{A}^1_*$-fibration such that $\mathrm{Sing}(p) \cong \mathbb{A}^1$ and the fibers are given as follows. Set $\alpha = xy$, $\beta = x^2(yz + 1)$. Then $p^{-1}(\alpha, \beta) = \{(\alpha y^{-1}, y, \alpha^{-2}(\beta y - \alpha^2 y^{-1}) \mid y \in \mathbb{C}^*\} \cong \mathbb{A}^1_*$ if $\alpha \ne 0$, $p^{-1}(0, \beta) = \{y = 0, \ x^2 = \beta\} \cong \mathbb{A}^1(x = \sqrt{\beta}, y = 0) \sqcup \mathbb{A}^1(x = -\sqrt{\beta}, y = 0)$ if $\alpha = 0$, $\beta \ne 0$ and $p^{-1}(0, 0) = \{x = 0\} \cong \mathbb{A}^2$ if $\alpha = \beta = 0$. $\square$

Winkelmann [47] kindly communicated us of the following example of a flat $\mathbb{A}^1_*$-fibration $p : \mathbb{A}^3 \to \mathbb{A}^2$ which is not surjective.

Proposition 2.16. *Let $f(x) = \prod_{i=1}^{n}(x - i)$ and let $p : \mathbb{A}^3 \to \mathbb{A}^2$ be the morphism defined by $(x, y, z) \mapsto (x + zx + zyf(x), x + yf(x))$. Then the following assertions hold.*

(1) *A general fiber is isomorphic to $\mathbb{A}^1_{n*}$, where $\mathbb{A}^1_{n*}$ is the affine line minus n points. Hence p is an $\mathbb{A}^1_*$-fibration if $n = 1$.*

(2) $\mathrm{Im}\,(p) = \mathbb{A}^2 \setminus S$, *where $S = \{(k, 0) \mid 1 \le k \le n\}$.*

(3) *p is equi-dimensional. Hence p is flat.*

(4) *The singular locus $\mathrm{Sing}(P)$, that is the locus of points P of the base $\mathbb{A}^2$ over which $p^{-1}(P) \not\cong \mathbb{A}^1_{n*}$, is the union of $\{(\alpha, 0) \mid \alpha \in \mathbb{C}\}$ and $\bigcup_{k=1}^{n}\{(\alpha, k) \mid \alpha \in \mathbb{C}\}$. Hence $\mathrm{Sing}(p)$ consists of disjoint $n + 1$ affine lines. For any $\alpha \in \mathbb{C} - \{1, \ldots, n\}$, the fiber $p^{-1}(\alpha, 0) \cong \mathbb{A}^1$. For any $k \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{C}$, the fiber $p^{-1}(\alpha, k) \cong \mathbb{A}^1 \cup \mathbb{A}^1_{(n-1)*}$.*

Proof. (1) Set $\alpha = x + z(x + yf(x))$ and $\beta = x + yf(x)$. Then $\alpha = x + z\beta$. Hence if $\beta \ne 0$, we have $z = (\alpha - x)/\beta$. Further, if $\beta \notin \{1, 2, \ldots, n\}$, then the equation $\beta = x + yf(x)$ is solved as $y = (\beta - x)/f(x)$. In fact, if (x, y) is a solution of $\beta = x + yf(x)$ and $f(x) = 0$, then $x = \beta$ and $\beta \in \{1, 2, \ldots, n\}$, a contradiction. Hence $p^{-1}(\alpha, \beta) \cong \mathbb{A}^1 - \{1, 2, \ldots, n\} \cong \mathbb{A}^1_{n*}$.

(2) Suppose $\beta = 0$. Then $x = \alpha$ and z is free if $\alpha = x + z\beta$ has a solution. Further, $0 = \alpha + yf(\alpha)$, which implies $\alpha = 0$ if $f(\alpha) = 0$. But this is impossible. So, $p^{-1}(k, 0) = \emptyset$ if $k \in \{1, 2, \ldots, n\}$. If $f(\alpha) \ne 0$, y is solved as $y = -\alpha/f(\alpha)$. So, $p^{-1}(\alpha, 0) = \{(\alpha, -\alpha/f(\alpha), z) \mid z \in \mathbb{C}\} \cong \mathbb{A}^1$. Suppose $\beta \in \{1, 2, \ldots, n\}$. Then $z = (\alpha - x)/\beta$ and the equation $x + yf(x) = \beta$ is written as $(x - \beta)(yg(x) + 1) = 0$, where $f(x) = g(x)(x - \beta)$. If $x = \beta$ then y is free. If $yg(x) + 1 = 0$ then $y = -1/g(x)$. This implies that $p^{-1}(\alpha, \beta) = \mathbb{A}^1 \cup \mathbb{A}^1_{(n-1)*}$ with two components meeting in a single point $(\beta, -1/g(\beta), (\alpha - \beta)/\beta)$ in the (x, y, z)-coordinates. By the above reasoning, we know that $\mathrm{Im}\,(p) = \mathbb{A}^2 \setminus S$ as stated above. The assertions (3) and (4) are also shown in the above argument. $\square$

If we restrict the morphism $p : \mathbb{A}^3 \to \mathbb{A}^2$ in the above proposition to a general linear plane in $\mathbb{A}^3$, we obtain an example of an endomorphism $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ which is not surjective. Such examples have been constructed by Jelonek [14].

We further study the singular fibers of $\mathbb{A}^1_*$-fibrations.

EXAMPLE **2.17.** Let $\widetilde{B} = \mathbb{C}[x, y, z, z^{-1}]$ be the Laurent polynomial ring in z over a polynomial ring $\widetilde{A} = \mathbb{C}[x, y]$. Let G_m and a cyclic group G of order n act on $\widetilde{B}$ by

$$^t(x, y, z) = (x, y, tz) \quad \text{and} \quad {}^\zeta(x, y, z) = (x, \zeta y, \zeta^d z),$$

where ζ is a primitive n-th root of 1, G is identified with the subgroup $\langle \zeta \rangle$ of G_m generated by ζ and d is a positive integer with $\gcd(d, n) = 1$. Then the following assertions hold.

(1) Let B (resp. A) be the G-invariant subring of $\widetilde{B}$ (resp. $\widetilde{A}$) and let d' be a positive integer such that $dd' \equiv 1 \pmod{n}$. Then $B = \mathbb{C}[x, y^n, y/z^{d'}, z^n, z^{-n}] = \mathbb{C}[x, \eta, U, T, T^{-1}]/(U^n = \eta T^{-d'})$ and $A = \mathbb{C}[x, \eta]$, where $\eta = y^n$, $U = y/z^{d'}$ and $T = z^n$.

(2) Let $Y = \operatorname{Spec} B$, $X = \operatorname{Spec}(A)$ and $q : Y \to X$ be the morphism induced by the inclusion $A \hookrightarrow B$. Then q is the quotient morphism of Y by the G_m-action given by $^t(x, \eta, U, T) = (x, \eta, t^{-d'}U, t^n T)$.

(3) The singular locus $\operatorname{Sing}(q)$ is the line $\eta = 0$ on X, and the singular fibers over $\operatorname{Sing}(q)$ are of the form $n\mathbb{A}_*^1$.

$\square$

In the above example, the G-action on $\widetilde{B}$ commutes with the G_m-action. Hence the G_m-action descends onto Y. Write $dd' = 1 + cn$. Then $^t(U^d T^c) = t^{-1}(U^d T^c)$. Hence the isotropy subgroup is trivial (resp. G) if $U \neq 0$ (resp. $U = 0$).

Lemma 2.18. *Let $p : Y \to X$ be an $\mathbb{A}_*^1$-fibration satisfying the following conditions.*

(1) *Y and X are smooth, p is equi-dimensional and $\operatorname{Sing}(p)$ is a smooth irreducible curve, say C.*

(2) *There exists a positive integer $n > 1$ such that the fiber F_P over every point $P \in C$ is of the form $n\mathbb{A}_*^1$.*

(3) *Either X is factorial, or there exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(-C)$.*

Then there exists a cyclic covering $\mu : \widetilde{X} \to X$ of order n ramifying totally over C such that the normalization $\widetilde{Y}$ of $Y \times_X \widetilde{X}$ is an $\mathbb{A}_*^1$-bundle over $\widetilde{X}$, where $\nu : \widetilde{Y} \to Y \times_X \widetilde{X}$ is the normalization morphism. Further, there exists a G_m-action on $\widetilde{Y}$ such that the projection $\widetilde{q} : \widetilde{Y} \xrightarrow{\mu} Y \times_X \widetilde{X} \to \widetilde{X}$ is the quotient morphism by the G_m-action. The quotient morphism $\widetilde{q}$ commutes with the cyclic covering group G, and hence descends down to the quotient morphism $q : Y \to X$ by the induced G_m-action which coincides with the given $\mathbb{A}_*^1$-fibration.

Proof. Write $X = \operatorname{Spec} A$. If X is factorial, let $f \in A$ define the curve C and let $\widetilde{X} = \operatorname{Spec} A[\xi]/(\xi^n - f)$. If $\mathcal{O}_X(-C) \cong \mathcal{L}^{\otimes n}$, let $\widetilde{X} = \mathcal{S}pec_X \bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes i}$. Then the morphism $\mu : \widetilde{X} \to X$ is a cyclic covering of order n totally ramifying over the curve C.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of $p^{-1}(C)$ and let η_i be an element of $\Gamma(U_i, \mathcal{O}_Y)$ such that $\eta_i = 0$ defines $p^{-1}(C)_{\mathrm{red}}|_{U_i}$ for each i. We consider the case where C is defined by $f = 0$. The other case can be treated in a similar fashion. We can write $f = u_i \eta_i^n$ for $u_i \in \Gamma(U_i, \mathcal{O}_Y^*)$. Then $\widetilde{Y}$ over U_i is defined as $\mathcal{S}pec_{U_i} \mathcal{O}_{U_i}[\xi/\eta_i]$, where $(\xi/\eta_i)^n = u_i$. This implies that the morphism $\rho : \widetilde{Y} \xrightarrow{\nu} Y \times_X \widetilde{X} \xrightarrow{p_1} Y$ with the first projection p_1 is a finite étale morphism, whence $\widetilde{Y}$ is smooth. It is clear that $\widetilde{q} : \widetilde{Y} \to \widetilde{X}$ is an $\mathbb{A}^1_*$-fibration. Since ρ is finite and étale, $\widetilde{q}^{-1}(\widetilde{Q})$ with $\widetilde{Q} \in \mu^{-1}(C)_{\mathrm{red}}$ is a reduced $\mathbb{A}^1_*$ or a disjoint union of reduced $\mathbb{A}^1_*$'s. Let D be a general smooth curve on $\widetilde{X}$ passing through $\widetilde{Q}$ and let $\widetilde{Z}$ be the normalization of $\widetilde{q}^{-1}(D)$. Then the canonical morphism $\overline{q} : \widetilde{Z} \to D$ is an $\mathbb{A}^1_*$-fibration on a normal affine surface $\widetilde{Z}$ and the fiber $\overline{q}^{-1}(\widetilde{Q})$ maps surjectively onto the fiber $\widetilde{q}^{-1}(\widetilde{Q})$. By Lemma 2.2, $\overline{q}^{-1}(\widetilde{D})$ does not have two or more irreducible components which surject onto $\mathbb{A}^1_*$. Hence $\widetilde{q}^{-1}(\widetilde{Q})$ consists of a single reduced component isomorphic to $\mathbb{A}^1_*$. This implies that $\widetilde{Y}$ is an $\mathbb{A}^1_*$-bundle over $\widetilde{X}$.

An $\mathbb{A}^1_*$-bundle has a standard G_m-action along the fibers. Namely, if $\widetilde{q}^{-1}(V_j) \cong V_j \times \operatorname{Spec} \mathbb{C}[\tau_j, \tau_j^{-1}]$ for an open covering $\mathcal{V} = \{V_j\}_{j \in J}$ of $\widetilde{X}$, then G_m acts as ${}^t\tau_j = t\tau_j$. This action clearly commutes with the action of the cyclic covering group G. Note that $\widetilde{Y}/G \cong Y$. Hence the G_m-action on $\widetilde{Y}$ descends down to Y, and gives rise to a G_m-action on Y. The quotient morphism $q : Y \to Y/\!/G_m$ coincides with $p : Y \to X$ because $q = p$ on the open set $p^{-1}(X \setminus C)$. $\qquad\square$

A G_m-action of a smooth affine threefold is called *equi-dimensional* if the quotient morphism $q : Y \to X$ is equi-dimensional.

Lemma 2.19. *Let Y be a smooth affine threefold with a G_m-action and let $q : Y \to X$ be the quotient morphism with $X = Y/\!/G_m$. Suppose that q has equi-dimension one, q is an $\mathbb{A}^1_*$-fibration and X is smooth. Then a singular fiber of q is either a tube or a cross.*

Proof. Let F_P be a singular fiber F of q. Let C be a smooth irreducible curve on X such that $P \in C$. Let $Z = Y \times_X C$ and let $\nu : \widetilde{Z} \to Z$ be the normalization morphism. Then the projection $\rho : Z \to C$ and $\widetilde{\rho} = \rho \cdot \nu : \widetilde{Z} \to C$ are the quotient morphisms by the induced G_m-actions on Z and $\widetilde{Z}$. The fiber $\rho^{-1}(P)$ is the fiber F_P

and is the image of $\widetilde{F}_P = \widetilde{\rho}^{-1}(P)$ by ν. By Lemma 2.2, the fiber $\widetilde{F}_P$ is either a tube or a cross. If $\widetilde{F}_P$ is a tube, then F_P is also a tube because ν restricted onto $\widetilde{F}_P$ commutes with the G_m-action. Suppose that $\widetilde{F}_P$ is a cross. Since the intersection point $\widetilde{Q}$ of two affine lines of $\widetilde{F}_P$ is a fixed point and $\nu_{\widetilde{F}_P}$ is surjective, either F_P consists of two components meeting in a point $Q = \nu(\widetilde{Q})$ or F_P is a contractible curve with Q a fixed point and $F_P - \{Q\}$ a G_m-orbit. In the latter case, two branches of a cross are folded into a single curve. But this is impossible because the G_m-action on a cross viewed near the point $\widetilde{Q}$ has weights $-a, b$ respectively on two branches with $ab > 0$. So, F_P consists of two branches meeting in one point Q. Consider the induced representation of G_m on the tangent space $T_{Y,Q}$ at the fixed point Q. We can write it as $^t(x, y, z) = (x, t^{-a}y, t^b z)$ for a suitable system of local coordinates $\{x, y, z\}$. Then the fiber F_P is given by $xy = 0$ locally at Q. Hence two branches of F_P meet transversally at Q, and F_P is a cross. $\square$

We can prove a *converse* of this result.

Theorem 2.20. *Let $p : Y \to X$ be an $\mathbb{A}^1_*$-fibration on a smooth affine threefold Y. Suppose that X is normal, p is equi-dimensional and the singular fibers are tubes or crosses over the points of $\mathrm{Sing}(p)$ except for a finite set of points. Then there exists an equi-dimensional G_m-action on Y such that the quotient morphism $q : Y \to Y /\!/ G_m$ coincides with the given $\mathbb{A}^1_*$-fibration p. Hence X has at worst cyclic quotient singularities.*

Proof. Since $\mathrm{Sing}(X)$ is a finite set, if a G_m-action is constructed on $p^{-1}(X \setminus \mathrm{Sing}(X))$ in such a way that the quotient morphism coincides with p restricted on $p^{-1}(X \setminus \mathrm{Sing}(X))$, then the G_m-action extends to Y by Hartog's theorem and the quotient morphism coincides with p. Since, as shown below, the construction of a G_m-action is local over $X \setminus \mathrm{Sing}(X)$, we may restrict ourselves to an affine open set of $X \setminus \mathrm{Sing}(X)$ and assume that X is smooth from the beginning.

By Lemmas 2.9 and 2.10, the singular locus $\mathrm{Sing}(p)$ has no isolated points. Let $\mathrm{Sing}(p) = C_1 \cup \cdots \cup C_r$ be the irreducible decomposition. Let

$$X_i = X \setminus (C_1 \cup \cdots \cup \overset{\vee}{C}_i \cup \cdots \cup C_r)$$

for $1 \le i \le r$, let $Y_i = p^{-1}(X_i)$ and let $p_i = p|_{Y_i} : Y_i \to X_i$. Then Y_i and X_i are affine and the $\mathbb{A}^1_*$-fibration p_i has an irreducible singular locus, say C_i by abuse of the notation. If C_i has singular points, we replace Y_i and X_i first by $p_i^{-1}(X \setminus \mathrm{Sing}(C_i))$ and $X \setminus \mathrm{Sing}(C_i)$ respectively and then by $p_i^{-1}(U_{i\lambda})$ and $U_{i\lambda}$ respectively, where $\{U_{i\lambda}\}_{\lambda \in \Lambda}$ is an affine open

covering of X_i. If there exist a family of equi-dimensional G_m-action $\sigma_i : G_m \times Y_i \to Y_i$ which induce the standard multiplication $\tau \mapsto t\tau$ for a variable on a general fiber τ, then the actions $\{\sigma_i\}_{i=1}^r$ patch together and define an equi-dimensional G_m-action $\sigma : G_m \times Y \to X$. By Hartog's theorem, this G_m-action is also extended over the fibers lying over the singular points of $\mathrm{Sing}(p)$. In fact, the union of the fibers over the singular points of $\mathrm{Sing}(p)$ has codimension greater than one.

So, we assume that $\mathrm{Sing}(p)$ is an irreducible smooth curve C. By shrinking X again to a smaller open set, we may assume that the curve C is principal, i.e., defined by a single equation $f = 0$. If the singular fibers over the points of C are tubes, the existence of a G_m-action follows from Lemma 2.18. Suppose that the fibers over C are crosses. Then $p^{-1}(C) = E_1 \cup E_2$ with irreducible component E_1, E_2. Then $Y \setminus E_1$ and $Y \setminus E_2$ are affine and the fibration p restricted to $Y \setminus E_1$ and $Y \setminus E_2$ have tubes over the curve C. Hence Lemma 2.18 implies that there exist G_m-actions on $Y \setminus E_1$ and $Y \setminus E_2$ and that they coincides on the general fibers of p. Hence they patch together and define a G_m-action on Y.

The last assertion follows from Lemma 2.13. $\qquad\qquad\square$

For examples with tubes or crosses as the singular fibers, we refer to Example 2.17 for tubes and Lemmas 4.1 and 4.4 below for crosses. In view of Lemma 2.2, we have a satisfactory description on singular fibers of $\mathbb{A}^1_*$-fibrations on normal affine surfaces. If Z is a normal affine surface with an $\mathbb{A}^1_*$-fibration $\rho : Z \to C$, then $p = \rho \times C' : Z \times C' \to C \times C'$ gives an $\mathbb{A}^1_*$-fibration on a normal affine threefold $Z \times C'$, where C' is a smooth affine curve. Hence the same singular fibers as in the surface case appear in the product threefold case. But we can say much less in general. Let $p : Y \to X$ be an $\mathbb{A}^1_*$-fibration, where Y is a smooth affine threefold and X is a smooth affine surface. Let F_P be the singular fiber of p lying over a point $P \in X$. Let C be a smooth curve on X through P and let Z be the normalization of $Y \times_X C$. Then the induced morphism $p_C : Z \to C$ is an $\mathbb{A}^1_*$-fibration. Hence the fiber F_P has a finite covering $\widetilde{F}_P \to F_P$, where $\widetilde{F}_P$ is a fiber of p_C and hence has the form as described in Lemma 2.2. We do not know exactly what the singular fiber F_P itself looks like.

Concerning the coexistence of tubes and crosses in the quotient morphism $q : Y \to X$ by a G_m-action, we have the following result.

Lemma 2.21. *Let Y be a homology threefold with an effective, equi-dimensional G_m-action and let $q : Y \to X$ be the quotient morphism. Suppose that q is an $\mathbb{A}^1_*$-fibration, X is smooth and $\dim Y^{G_m} = 1$. Then there are no tubes as singular fibers of q.*

Proof.   By the assertion (3) of Theorem 2.6, $Y^{G_m} \cong \mathbb{A}^1$ and $q|_{Y^{G_m}}$: $Y^{G_m} \to X$ is a closed embedding. By Lemma 2.19, any fiber through a point of Y^{G_m} is a cross. Let $F = F_P$ be a tube of multiplicity $m > 1$. Let p be a prime factor of m and let $\Gamma = \mathbb{Z}/p\mathbb{Z}$. Consider the induced Γ-action on Y as Γ is a subgroup of G_m. Then F is contained in the Γ-fixed point locus Y^Γ (see the argument after the proof of Lemma 2.13). Let Q be a point on F. Then the induced tangential representation of Γ on $T_{Y,Q}$ is written as

$$\zeta(x, y, z) = (x, \zeta^a y, \zeta^b z), \ 0 \le a < p, \ 1 \le b < p,$$

where ζ is a primitive p-th root of unity and x is a coordinate with the tangential direction of the fiber. If $a > 0$, $b > 0$, then the point $P = q(Q)$ is a singular point of X (see the argument after the proof of Lemma 2.13). Hence $a = 0$. This implies that the component of Y^Γ containing F has dimension two. By the Smith theory, the locus Y^Γ is a connected closed set. Hence there is a point Q' of Y^{G_m} such that the fiber $F'_{P'}$ through Q' is contained in Y^Γ. Then we can write $F'_{P'} = m_1 \mathbb{A}^1 + m_2 \mathbb{A}^2$ with $p \mid m_i$ for $i = 1, 2$. But this is impossible. In fact, let C be a smooth curve on X passing through the point P' and set $Z = Y \times_X C$. Then Z has an induced G_m-action. Let $\widetilde{Z}$ be the normalization of $Z \times_C \widetilde{C}$, where $\widetilde{C} \to C$ is a finite covering of order p totally ramifying over the point P'. Then there is an induced G_m-action on $\widetilde{Z}$ such that the induced morphism $\widetilde{q} : \widetilde{Z} \to \widetilde{C}$ is the quotient morphism. The fiber $\widetilde{q}^{-1}(\widetilde{P}')$ with $\widetilde{P}'$ a point of $\widetilde{C}$ lying over P' consists of p-copies of the cross $\mathbb{A}^1 + \mathbb{A}^1$ when taken with reduced structure. But this is impossible.   □

Finally in this section, we discuss Question 2.12. In fact, we prove a more general result.

Lemma 2.22. *Let $p : Y \to X$ be a dominant morphism from an algebraic variety of dimension $n+r$ to an algebraic variety X of dimension n. Let P be a point of X and let Q be a point of Y such that $p(Q) = P$. Suppose that Y is smooth at Q and the fiber $F := p^{-1}(P)$ is reduced, r-dimensional and smooth at Q. Then P is smooth at X.*

Proof.   Let $(R, \mathfrak{m})$ (resp. $(S, \mathfrak{M})$) be the local ring of X (resp. Y) at P (resp. Q). Since the fiber F is defined by $\mathfrak{m}S$ at Q and since $S/\mathfrak{m}S$ is a regular local ring as Q is a smooth point of F, it follows that $\mathfrak{M} = \mathfrak{m}S + (z_{n+1}, \ldots, z_{n+r})S$ for elements $z_{n+1}, \ldots, z_{n+r} \in \mathfrak{M}$. Since $(S, \mathfrak{M})$ is a regular local ring of dimension $n + r$, we find n elements $z'_1, \ldots, z'_n$ of $\mathfrak{m}$ with images $z_1, \ldots, z_n$ in S such that $\{z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+r}\}$ is a regular system of parameters of $(S, \mathfrak{M})$. Since the completion

$(\widehat{S}, \widehat{\mathfrak{M}})$ is isomorphic to $\mathbb{C}[[z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+r}]]$, we can express any $h \in \widehat{S}$ as a formal power series in $z_{n+1}, \ldots, z_{n+r}$ with coefficients in $\mathbb{C}[[z_1, \ldots, z_n]]$,

$$h = \sum_{\vec{i}=0}^{\infty} \alpha_{\vec{i}} z^{\vec{i}}, \ \alpha_{\vec{i}} \in \mathbb{C}[[z_1, \ldots, z_n]],$$

where $z^{\vec{i}} = z_{h+1}^{i_{n+1}} \cdots z_{n+r}^{i_{n+r}}$ for $\vec{i} = (i_{n+1}, \ldots, i_{n+r})$. We shall show that the completion $(\widehat{R}, \widehat{\mathfrak{m}})$ is isomorphic to the formal power series ring $\mathbb{C}[[z_1, \ldots, z_n]]$. We consider the local complete intersection

$$H = \{z_j = 0 \mid n+1 \le j \le n+r\}$$

in Y near the point Q as a section transversal to the fiber F at the point Q. We may assume that the restriction $p|_H : H \to X$ is quasi-finite. Hence the injective local ring homomorphism $R \to S \to S/zS$ induces an injective local homomorphism

$$(\widehat{R}, \widehat{\mathfrak{m}}) \to (\widehat{S}, \widehat{\mathfrak{M}}) \to (\widehat{S}/z\widehat{S}, \widehat{\mathfrak{M}}/z\widehat{S}),$$

because $(\widehat{S}/(z_{n+1}, \ldots, z_{n+r})\widehat{S}, \widehat{\mathfrak{M}}/(z_{n+1}, \ldots, z_{n+r})\widehat{S})$ is viewed as the completion of the local ring $\mathcal{O}_{H,Q}$. Sending z_i to z_i', $i = 1, \ldots, n$, we obtain a homomorphism

$$\mathbb{C}[[z_1, \ldots, z_n]] \longrightarrow \widehat{R},$$

which gives an isomorphism when composed with the mapping $\widehat{R} \to (\widehat{S}/z\widehat{S})$. Let h' be an element of $\widehat{R}$ and h its image in $\widehat{S}$. With h expressed as a power series as above, we find that h' and the image in $\widehat{R}$ of $\alpha_{\vec{0}} \in \mathbb{C}[[z_1, \ldots, z_n]]$ have the same image in $\widehat{\mathfrak{M}}/z\widehat{S}$ and hence coincide. Hence $\widehat{R} \cong \mathbb{C}[[z_1, \ldots, z_n]]$. This implies that X is smooth at P. $\qquad \square$

3. Homology threefolds with $\mathbb{A}^1$-fibrations

In [9], it is shown that an $\mathbb{A}^1$-fibration $p : Y \to X$ from a smooth affine threefold to a normal surface has a factorization

$$p : Y \xrightarrow{q} \widetilde{X} \xrightarrow{\sigma} X,$$

where $q : Y \to \widetilde{X}$ is the quotient morphism by a G_a-action and $\sigma : \widetilde{X} \to X$ is the birational morphism such that $\widetilde{A} = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is the factorial closure of $A = \Gamma(X, \mathcal{O}_X)$ in $B = \Gamma(Y, \mathcal{O}_Y)$. Namely,

$$\widetilde{A} = \{b \in B \mid b \text{ is a factor of an element } a \in A\}.$$

Thus we may look into the quotient morphism $q : Y \to X$ by a G_a-action. We set $\mathrm{Sing}(q) = \{P \in X \mid F_P \not\cong \mathbb{A}^1\}$, where F_P is the fiber over $P \in X$, and call it the *singular locus* of q.

Lemma 3.1. *Let $q : Y \to X$ be the quotient morphism of a smooth affine threefold Y with respect to a G_a-action. Suppose that q is equi-dimensional. Then the following assertions hold.*

 (1) *If a fiber F_P has an irreducible component which is reduced in F_P, then the point P is smooth in X.*
 (2) $\mathrm{Sing}(q)$ *is a closed set.*

Proof.   (1) The assertion follows from Lemma 2.22.

(2) Since q has equi-dimension one, the fixed point locus Y^{G_a} consists of fiber components [9, Corollary 3.2]. Then every fiber of q is a disjoint union of contractible curves [9, Lemma 3.5]. Further, a contractible irreducible component is isomorphic to $\mathbb{A}^1$ if it contains a non-fixed point, or is contained in Y^{G_a}.

It suffices to show that given a fiber F_P isomorphic to $\mathbb{A}^1$ there exists an open neighborhood U of P in X with $F_{P'} \cong \mathbb{A}^1$ for all $P' \in U$. The subsequent proof is similar to the one for G_m-actions (cf. Lemma 2.11). Let C be an irreducible curve on X through P and let Z be the normalization of $Y \times_X C$. Suppose that the fiber of q over a general point of C consists of m copies of $\mathbb{A}^1$, where $m > 1$. Then Z has an induced G_a-action such that

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \mu\ } & Y \times_X C \\
\downarrow{\scriptstyle \widetilde{q}} & & \downarrow{\scriptstyle q} \\
\widetilde{C} & \xrightarrow{\ \nu\ } & C
\end{array}
$$

where $\widetilde{q}$ is the quotient morphism and $\widetilde{C}$ is the normalization of C in Z. The morphism $\nu : \widetilde{C} \to C$ is a finite covering of degree m (the Stein factorization), which is ramified over the point P. It then follows that the fiber F_P is non-reduced, and this is a contradiction.

Consider the closure $\overline{\mathrm{Sing}(q)}$ and remove from X all the irreducible components of $\overline{\mathrm{Sing}(q)}$ over which a general fiber of q consists of more than one irreducible component. Thus we may assume that all singular fibers of q are of the form $m\mathbb{A}^1$ with $m > 1$. Note that P is a smooth point by the assertion (1). Replacing X by an affine open neighborhood U of P and accordingly Y by the inverse image $q^{-1}(U)$, we may assume that X is smooth. Consider the tangential homomorphism of the tangent bundles

$$dq : \mathcal{T}_Y \to q^*\mathcal{T}_X$$

and let $\mathcal{C}$ be the cokernel of dq. Then $\mathcal{C}$ is a coherent $\mathcal{O}_Y$-module. The support $T = \mathrm{Supp}\,(\mathcal{C})$ is a closed set such that $T = q^{-1}(q(T))$. If $P \in \overline{\mathrm{Sing}(q)}$, then $F_P \subset T$, but $F_P \cap T = \emptyset$. This is a contradiction. Then, by Dutta [3, Theorem], there exists an open neighborhood U of P such that $q^{-1}(U)$ is an $\mathbb{A}^1$-bundle over U. Hence $\mathrm{Sing}(q)$ is a closed set. $\square$

Concerning a G_a-action, we ask the following

Question 3.2. Let Y be an affine variety with a G_a-action. Suppose that the algebraic quotient $X = Y /\!/ G_a$ exists as an affine variety, i.e., the G_a-invariant subring of $\Gamma(Y, \mathcal{O}_Y)$ is an affine domain. Let Q be a point of Y with trivial isotropy group, i.e., a point which is not G_a-fixed. Is then the fiber F_P of the quotient morphism $q : Y \to X$ passing through the point Q reduced, where $P = q(Q)$?

The answer is negative and an example is given by an affine pseudo-plane [9, Corollary 3.16]. See also [39, Lemma 2.4].

EXAMPLE **3.3.** Let C be the smooth conic in $\mathbb{P}^2 = \mathrm{Proj}\,\mathbb{C}[X_0, X_1, X_2]$ defined by $X_2^2 = X_0 X_1$. Let L be the tangent line of C at the point $(X_0, X_1, X_2) = (0, 1, 0)$. Hence L is defined by $X_0 = 0$. Let Λ be the linear pencil generated by C and $2L$. Then the G_a-action on $\mathbb{P}^2$ defined by

$$^tX_0 = X_0, \ ^tX_1 = X_1 + 2X_2 t + X_0 t^2, \ ^tX_2 = X_2 + X_0 t$$

has a unique fixed point Q and preserves the members of the pencil Λ. Let X be the complement of C in $\mathbb{P}^2$. Then Λ defines an $\mathbb{A}^1$-fibration $q : X \to \mathbb{A}^1$ which turns out to be the quotient morphism of the induced G_a-action on X. Although the G_a-action has no fixed points, q has a multiple fiber 2ℓ which comes from the member $2L$ of Λ. The affine surface X is an affine pseudo-plane and its universal covering is a Danielewski surface.

This is the case for every affine pseudo-plane $q : X \to Z$ of type (m, r) with an integer $r \geq 1$ such that $r \equiv 1 \pmod{m}$. For the definition of an affine pseudo-plane of type (m, r), see [31] where the type is denoted by (d, r) instead of (m, r). In particular, $q : X \to Z$ is an $\mathbb{A}^1$-fibration with a unique singular fiber F_0 of multiplicity $m > 1$, i.e., $F_0 = m\mathbb{A}^1$ and $Z \cong \mathbb{A}^1$. X is denoted by $X(m, r)$.

Lemma 3.4. *Let $q : X \to Z$ be an affine pseudo-plane of type (m, r) with $r \equiv 1 \pmod{m}$. Then q is given by a fixed-point free G_a-action.*

Proof. Let $\widetilde{X}(m, r)$ be the universal covering of $X(m, r)$. Let $H(m) = \mathbb{Z}/m\mathbb{Z}$ be the covering group which is identified with the m-th roots of

unity. By [31, Lemma 2.6], $\widetilde{X}(m,r)$ is isomorphic to a hypersurface in $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x,y,z]$ defined by

$$x^r z + (y^m + a_1 x y^{m-1} + \cdots + a_{m-1} x^{m-1} y + a_m x^m) = 1, \ a_i \in \mathbb{C}.$$

The Galois group $H(m)$ acts as

$$^\lambda(x,y,z) = (\lambda x, \lambda y, \lambda^{-r} z), \ \lambda \in H(m).$$

The projection $(x,y,z) \mapsto x$ defines an $\mathbb{A}^1$-fibration $\widetilde{q} : \widetilde{X}(m,r) \to \widetilde{Z}$, where $\widetilde{Z} \cong \mathbb{A}^1$. Further, there is a G_a-action on $\widetilde{X}(m,r)$ defined by

$$^t(x,y,z) =$$
$$(x, y+tx^r, z - x^{-r}\{((y+tx^r)^m + a_1 x(y+tx^r)^{m-1} + \cdots + a_m x^m)$$
$$-(y^m + a_1 x y^{m-1} + \cdots + a_{m-1} x^{m-1} y + a_m x^m)\}).$$

Then it follows that

(1) the fiber $\widetilde{F}_0$ of $\widetilde{q}$ over the point $x = 0$ is a disjoint union of m-copies of the affine line,
(2) the G_a-action preserves the fibration $\widetilde{q}$,
(3) the G_a-action preserves and acts non-trivially on each connected component of $\widetilde{F}_0$,
(4) if $r \equiv 1 \pmod{m}$ then the $H(m)$-action commutes with the G_a-action.

Hence the G_a-action descends to a G_a-action on $X(m,r)$ which has no fixed points. □

We consider, however, a result implied by the assumptions in Question 3.2. We need some auxiliary results. Let $B = \Gamma(Y, \mathcal{O}_Y)$ and let $A = \Gamma(X, \mathcal{O}_X)$. Let δ be a locally nilpotent derivation on B which corresponds to the given G_a-action. Further, let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point P. First we recall the following result in [39, Theorem 3.3].

Lemma 3.5. *Let B be an affine domain over $\mathbb{C}$ and let δ be a locally nilpotent derivation on B. Let $A = \operatorname{Ker} \delta$. Suppose that $B/\mathfrak{m}B$ is an integral domain over $\mathbb{C}$ of dimension one and that the associated G_a-action on $\operatorname{Spec} B/\mathfrak{m}B$ has no fixed points. Then the following assertions hold.*

(1) *For any integer $n > 0$, there exists an element $z_n \in B$ such that $B/\mathfrak{m}^n B = R_n[z_n]$, where R_n is an Artin local ring and we denote the residue class of z_n in $B/\mathfrak{m}^n B$ by the same letter.*
(2) *For $m > n$, we have a natural exact sequence*

$$0 \to \mathfrak{m}^n R_m \to R_m \xrightarrow{\theta_{nm}} R_n \to 0,$$

where θ_{nm} is a local homomorphism. Let $\widehat{R} = \varprojlim_n R_n$. Then $\widehat{R}$ is a complete local ring.

Let $\mathcal{O}$ be the local ring $\mathcal{O}_{X,P}$ and denote the ideal $\mathfrak{m}\mathcal{O}$ by the same letter $\mathfrak{m}$. Let $\widehat{\mathcal{O}}$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. Then $B \otimes_A \mathcal{O}$ is an affine domain over $\mathcal{O}$ with the associated locally nilpotent derivation δ such that δ is nonzero and $\operatorname{Ker} \delta = \mathcal{O}$. Further *we assume the following condition* (H) *that* $(\mathfrak{m}^n(B \otimes_A \mathcal{O})) \cap A = \mathfrak{m}^n$ *for every* $n > 0$. This condition is satisfied if $B \otimes_A \mathcal{O}$ is $\mathcal{O}$-flat (see [11, Lemma 1.4]).

Proposition 3.6. *With the above notations and assumptions, the following assertions hold.*

(1) $(B \otimes_A \mathcal{O})/\mathfrak{m}^n(B \otimes_A \mathcal{O}) \cong B/\mathfrak{m}^n B = R_n[z_n]$.

(2) *The projective limit* $\widehat{B} = \varprojlim_n (B \otimes_A \mathcal{O})/\mathfrak{m}^n(B \otimes_A \mathcal{O})$ *contains* $B \otimes_A \mathcal{O}$ *as a subring. Further,* $\widehat{B}$ *has a derivation* $\widehat{\delta}$ *and an element* $\widehat{z}$ *such that* $\widehat{\delta}|_{B\otimes_A\mathcal{O}} = \delta$, $\operatorname{Ker} \widehat{\delta} = \widehat{R}$ *and* $\widehat{\delta}(\widehat{z}) = 1$. *The ring* $\widehat{B}$ *itself is a subring of a formal power series ring* $\widehat{R}[[\widehat{z}]]$.

(3) *Let* $\widehat{B}_{\ln}$ *be the subring of* $\widehat{B}$ *consisting of elements for which* $\widehat{\delta}$ *is locally nilpotent. Then* $\widehat{B}_{\ln} = \widehat{R}[\widehat{z}]$ *and* $B \otimes_A \mathcal{O}$ *is a subring of* $\widehat{B}_{\ln}$.

Proof. (1) The assertion follows from Lemma 3.5.

(2) It suffices to show that if b is an element of $\bigcap_n \mathfrak{m}^n(B \otimes_A \mathcal{O})$ then $b = 0$. Suppose that $b \neq 0$. Then there exists an integer $r \geq 0$ such that $\delta^r(b) \neq 0$ and $\delta^{r+1}(b) = 0$. Then $\delta^r(b) \in \operatorname{Ker} \delta = \mathcal{O}$. Since $\delta(\mathfrak{m}^n(B\otimes_A\mathcal{O})) \subseteq \mathfrak{m}^n(B\otimes_A\mathcal{O})$ for every $n > 0$, by the condition (H), we have $\delta^r(b) \in (\bigcap_n \mathfrak{m}^n(B \otimes_A \mathcal{O})) \cap \mathcal{O} = \bigcap_n \mathfrak{m}^n$, which is zero by Krull's intersection theorem. Hence $\delta^r(b) = 0$. This is a contradiction. Hence $B \otimes_A \mathcal{O}$ is a subring of $\widehat{B}$. The sequence $\{z_n\}_{n\geq 1}$ is a Cauchy sequence because $z_m - z_n \in \mathfrak{m}^n(B \otimes_A \mathcal{O})$ for $m > n$. Hence it converges to an element $\widehat{z}$ of $\widehat{B}$. The derivation δ extends to a derivation $\widehat{\delta}$. Since we may assume that $\delta(z_n) \equiv 1 \pmod{\mathfrak{m}^n(B\otimes_A\mathcal{O})}$, it follows that $\widehat{\delta}(\widehat{z}) = 1$ in $\widehat{B}$. Let b_n be an element of $B \otimes_A \mathcal{O}$. Since $(B \otimes_A \mathcal{O})/\mathfrak{m}^n(B \otimes_A \mathcal{O}) \cong R_n[z_n]$ and $\widehat{z} - z_n \in \mathfrak{m}^n\widehat{B}$, there exists a polynomial $f_n(\widehat{z}) \in \widehat{R}[\widehat{z}]$ such that $b_n - f_n(\widehat{z}) \in \mathfrak{m}^n\widehat{B}$. This implies that a Cauchy sequence $\{b_n\}_{n\geq 0}$ in $B \otimes_A \mathcal{O}$ is approximated by a Cauchy sequence $\{f_n(\widehat{z})\}_{n\geq 0}$ in $\widehat{B}$. So, $\widehat{B}$ is a subring of $\widehat{R}[[\widehat{z}]]$.

(3) Let b be an element of $B \otimes_A \mathcal{O}$. Write

$$b = \alpha_0 + \alpha_1 \widehat{z} + \cdots + \alpha_i \widehat{z}^i + \cdots = \sum_{i=0}^{\infty} \alpha_i \widehat{z}^i$$

as an element of $\widehat{R}[[\widehat{z}]]$. If this is an infinite series, the module $M(b)$ generated over $\widehat{\mathcal{O}}$ by $\widehat{\delta^i}(b)$ for $i \geq 0$ is not finitely generated over $\widehat{\mathcal{O}}$ because it is not finitely generated over $\widehat{R}$. Meanwhile, $\widehat{\delta^i}(b) = \delta^i(b)$ for every $i \geq 0$ and $M(b)$ is finitely generated over $\widehat{\mathcal{O}}$ because δ is locally nilpotent. Hence $b \in \widehat{R}[\widehat{z}]$. This implies that $B \otimes_A \mathcal{O}$ is a subring of $\widehat{R}[\widehat{z}]$. It is clear that $\widehat{B}_{\mathrm{ln}} = \widehat{R}[\widehat{z}]$. $\qquad\square$

We recall the following result of Kaliman-Saveliev [18, Corollary 2.8].

Lemma 3.7. *Let Y be a smooth contractible affine threefold with a fixed-point free G_a-action. Then Y is isomorphic to $X \times \mathbb{A}^1$ with G_a acting on the second factor, where $X = Y//G_a$.*

This result inspires us a challenging problem.

Question 3.8. Let Y be a homology threefold with a fixed-point free G_a-action. Is Y G_a-equivariantly isomorphic to a product $X \times \mathbb{A}^1$ with G_a acting on the second factor, where $X = Y//G_a$?

If Y is replaced by a $\mathbb{Q}$-homology threefold, the answer is negative.

REMARK **3.9.** Let $q : X \to Z$ be an affine pseudo-plane as in Lemma 3.4. Then X is a $\mathbb{Q}$-homology plane. Let $Y = X \times \mathbb{A}^1$ with G_a acting on X and trivially on the second factor. Then Y is a $\mathbb{Q}$-homology threefold, $Y//G_a = Z \times \mathbb{A}^1$ and the quotient morphism is $q_Y = q \times \mathrm{id}_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to Z \times \mathbb{A}^1$. Hence $\mathrm{Sing}(q_Y) \cong \mathbb{A}^1$.

4. CONTRACTIBLE AFFINE THREEFOLDS WITH $\mathbb{A}^1_*$-FIBRATIONS

We first discuss Question 2.7.

Lemma 4.1. *Let X be a homology plane of log Kodaira dimension $-\infty$ or 1 and let C be a curve on X isomorphic to $\mathbb{A}^1$. Let $V = X \times \mathbb{A}^1$ with a G_m-action induced from the standard action on the $\mathbb{A}^1$-factor, i.e., ${}^t(Q, x) = (Q, tx)$, where $Q \in X$ and x is a coordinate of $\mathbb{A}^1$. Let $\sigma : W \to V$ be the blowing-up of V with center $C \times (0)$ which is identified with C. Let $Y = W \setminus \sigma'(X \times (0))$, where $\sigma'(X \times (0))$ is the proper transform of $X \times (0)$. Then the following assertions hold.*

    (1) *Y is a homology threefold with an induced G_m-action such that the quotient morphism $q : Y \to Y//G_m$ has the quotient space $Y//G_m \cong X$. If X is contractible, the threefold Y is also contractible.*

    (2) *The singular locus $\mathrm{Sing}(q)$ is the curve C. For every point P of C, the fiber $q^{-1}(P)$ consists of two affine lines meeting in one point. Thus we have the situation treated in Theorem 2.6, (3).*

(3) If $\overline{\kappa}(X) = 1$, then $\overline{\kappa}(Y) = 1$. Hence Y cannot be written as $Y \cong Z \times Y^{G_m}$, i.e., the answer to the question 2.7 is negative.

(4) If $\overline{\kappa}(X) = -\infty$, i.e., $X \cong \mathbb{A}^2$, then $Y \cong Z \times Y^{G_m}$ with Z a smooth affine surface with a G_m-action.

Proof. (1) Set $D = X \times (0)$, $L = C \times \mathbb{A}^1$, $L' = \sigma'(L)$ and $E' = \sigma^{-1}(C) \setminus \sigma'(D)$. Then $\sigma : (Y, E') \to (V, D)$ is an affine modification with $\sigma(E') = C$ (see [19]). Write $X = \operatorname{Spec} A$. Then $A[x]$ is the coordinate ring of V. Since A is factorial by Lemma 1.1, the curve C is defined by an element ξ of A. The hypersurface $D = X \times (0)$ in V is defined by $x = 0$. Let I be the ideal of $A[x]$ generated by ξ and x. Then Y has the coordinate ring $\Sigma_{I,x}(A[x])$ which is the affine modification of $A[x]$ along (x) with center I. Clearly, Y is a smooth affine threefold. By [19, Proposition 3.1 and Theorem 3.1], it follows that Y is a homology threefold and is contractible provided so is X. Let $q_0 : V \to X$ be the projection to X, which is in fact the quotient morphism by the G_m-action such that $V^{G_m} = D$. By the above process, the G_m-action is inherited on Y and the quotient morphism $q : Y \to Y//G_m$ is induced by q_0.

(2) Meanwhile, in passing from V to Y, the fiber over a point $Q \in X \setminus C$ loses the point $(Q, 0)$ in D and becomes isomorphic to $\mathbb{A}^1_*$. The fiber $q^{-1}(Q)$ is a cross $\mathbb{A}^1 + \mathbb{A}^1$ with two $\mathbb{A}^1$ meeting in the point $q^{-1} \cap (L' \cap E')$. Hence $E' \cap L' = Y^{G_m}$ and $\operatorname{Sing}(q) = C$.

(3) Suppose that $\overline{\kappa}(X) = 1$. In fact, there is a unique affine line lying on X. Since the general fibers of q are isomorphic to $\mathbb{A}^1_*$, we have an inequality $\overline{\kappa}(Y) \geq \overline{\kappa}(X) + \overline{\kappa}(F)$ by Kawamata [23], where F is a general fiber of q. Hence $\overline{\kappa}(Y) \geq 1$. Furthermore, X itself has an $\mathbb{A}^1_*$-fibration $\pi : X \to B$ such that C is a fiber of π and $\pi^{-1}(U) \cong U \times \mathbb{A}^1_*$, where U is an open set of B contained in $\mathbb{A}^1_{2*}$. By the above construction, $q^{-1}(\pi^{-1}(U)) \cong \pi^{-1}(U) \times \mathbb{A}^1_* \cong U \times \mathbb{A}^1_* \times \mathbb{A}^1_*$. Hence $\overline{\kappa}(Y) \leq \overline{\kappa}(U \times \mathbb{A}^1_* \times \mathbb{A}^1_*) = 1$. This implies that $\overline{\kappa}(Y) = 1$. If $Y \cong Z \times \mathbb{A}^1$ as inquired in Question 2.7, then it would follow that $\overline{\kappa}(Y) = -\infty$. But this is not the case.

(4) Suppose that $\overline{\kappa}(X) = -\infty$. Then $X \cong \mathbb{A}^2$, and the affine line C is chosen to be a coordinate line by AMS theorem. Namely, there exists a system of coordinates (ξ, η) of X such that C is defined by $\xi = 0$. The affine modification $\Sigma_{I,x}(A[x])$ is equal to $\mathbb{C}[x, \xi/x, \eta]$. Set $y = \xi/x$ and $R = \mathbb{C}[x, y]$. Then the induced G_m-action on $\Sigma_{I,x}(A[x])$ is given by $^t(x, y, \eta) = (tx, t^{-1}y, \eta)$. Hence the threefold Y is isomorphic to $Z \times \mathbb{A}^1$, where $Z = \operatorname{Spec} R \cong \mathbb{A}^2$ and $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[\eta]$. So, the answer to Question 2.7 is affirmative. $\square$

REMARK **4.2.** The quotient morphism $q : Y \to X$ has crosses $\mathbb{A}^1 + \mathbb{A}^1$ with each multiplicity one as the singular fibers over $\mathrm{Sing}(q) \cong \mathbb{A}^1$. The locus of intersection points of crosses is the fixed point locus $\Gamma = Y^{G_m}$. There are two embedded affine planes Z_1, Z_2 meeting transversally along Γ. If Y is a homology threefold, Z_1, Z_2 are defined by $f_1 = 0, f_2 = 0$. Hence Γ is defined by the ideal $I_1 = (f_1, f_2)$ of $B_1 = \Gamma(Y, \mathcal{O}_Y)$. The affine modification $B_2 = \Sigma_{I_1, f_1}(B)$ or $B_2' = \Sigma_{I, f_2}(B)$ gives rise to a smooth affine threefold $Y_2 = \mathrm{Spec}\, B_2$ or $Y_2' = \mathrm{Spec}\, B_2'$ with an equi-dimensional G_m-action, which gives the quotient morphism $q_2 : Y_2 \to X$ or $q_2' : Y_2' \to X$. Both Y_2 and Y_2' are homology threefolds (resp. contractible threefolds) if Y is a homology threefold (resp. contractible threefold). A difference between Y_2 (or Y_2') and Y is that the crosses have multiplicities. Namely, Y_2 (resp. Y_2') has crosses $2\mathbb{A}^1 + \mathbb{A}^1$ (resp. $\mathbb{A}^1 + 2\mathbb{A}^1$). We can repeat this process to produce homology threefolds or contractible threefolds which have crosses with higher multiplicities. If a cross is written as $m\mathbb{A}^1 + n\mathbb{A}^1$, then $\gcd(m, n) = 1$ (see the argument in the last part of the proof of Lemma 2.21).

Another remark to Lemma 4.1 is the following.

REMARK **4.3.** Take a $\mathbb{Q}$-homology plane X instead of a homology plane in the construction of Y in Lemma 4.1. Then we can consider such $\mathbb{Q}$-homology planes with log Kodaira dimension $-\infty, 0$ and 1. We take an embedded line C in X and blow up the center $C \times (0)$ in $V = X \times \mathbb{A}^1$. By the same construction, we obtain a smooth affine threefold Y with an equi-dimensional G_m-action. The threefold Y has the same homology groups as X. Hence Y is a $\mathbb{Q}$-homology threefold, which is not of the product type $Z \times \mathbb{A}^1$ with a $\mathbb{Q}$-homology plane Z provided $\bar{\kappa}(X) \neq -\infty$. For the embedded lines in the case of log Kodaira dimension 0, see [12, Theorem] for a complete classification.

In Lemma 4.1, the center of blowing-up is $C \times (0)$, where C is an embedded line in a homology plane, and the resulting homology threefold has log Kodaira dimension at most one. We can apply a process similar to the one used in Lemma 4.1 with a point as the center to obtain the first assertion of the following lemma. This is first constructed in [26, Example 3.7].

Lemma 4.4. (1) *Let X be a homology plane and let P_0 be a point of X. Let $Y_0 = X \times \mathbb{A}^1$ with G_m acting trivially on X and on $\mathbb{A}^1$ with weight -1. Let $Q_0 = P_0 \times (0)$. Blow up the point Q_0 and remove the proper transform of $X \times (0)$ to obtain a smooth affine threefold Y. Then Y is a homology threefold with a G_m-action such that the quotient morphism $q : Y \to X$ is induced by the first projection $p_1 : Y_0 \to X$, the fixed*

*point locus Y^{G_m} consists of the unique point which is the intersection
point of the proper transform of $P_0 \times \mathbb{A}^1$ with the exceptional surface of
the blowing-up (whence the G_m-action on Y is* hyperbolic*) and $\overline{\kappa}(Y) =
\overline{\kappa}(X)$. If X is contractible, so is Y.*

*(2) Let Y be a smooth homology threefold with a hyperbolic G_m-
action. Let Q_0 be the unique fixed point and let Z_0 be the two-dimensional
fiber component of the quotient morphism $q : Y \to X$, where $Q_0 \in Z_0$
and $X = Y//G_m$ (cf. Theorem 2.6). Let Y' be the affine modification
$\Sigma_{Q_0,Z_0}(Y)$ which is the blowing-up of Y at the center Q_0 with the proper
transform of Z_0 deleted off. Then Y' is a smooth homology threefold.
If Y is contactible, then so is Y'.*

The statement and the proof depend on [19, §3].

EXAMPLE **4.5.** Let Y be the Koras-Russell threefold $x + x^2y + z^2 + t^3 =
0$. Then a hyperbolic G_m-action on Y is given by

$$^\lambda(x,y,z,t) = (\lambda^6 x, \lambda^{-6}y, \lambda^3 z, \lambda^2 t), \ \lambda \in \mathbb{C}^*.$$

The fixed point Q_0 is $(0,0,0,0)$, and the two-dimensional fiber com-
ponent Z_0 is defined by $y = 0$. Let B be the coordinate ring of Y
and let $I = (x,y,z,t)$ which is the maximal ideal of Q_0. The affine
modification $B' = \Sigma_{I,y}(B)$, which is the coordinate ring of Y', is given
as

$$B' = \mathbb{C}[x',y,z',t'], \ x' = \frac{x}{y}, \ z' = \frac{z}{y}, \ t' = \frac{t}{y}.$$

Hence Y' is a hypersurface $x' + x'^2 y^2 + z'^2 y + t'^3 y^2 = 0$ and the hyperbolic
G_m-action is given by

$$^\lambda(x',y,z',t') = (\lambda^{12}x', \lambda^{-6}y, \lambda^9 z', \lambda^8 t'), \ \lambda \in \mathbb{C}^*.$$

We can further repeat the affine modifications of the same kind to
Y', Y'' etc.

The above Koras-Russell threefold and its affine modifications are
examples of smooth contractible threefolds Y with a hyperbolic G_m-
action such that the quotient X is isomorphic to that of the tangent
space T_{Q_0} at the unique fixed point Q_0 of Y by the induced tangential
representation. In [26, Theorem 4.1] a description of all such threefolds
is given.

In [17] the Makar-Limanov invariants $\mathrm{ML}(Y)$ of such threefolds are
computed. To apply this result the equation for Y' has to be brought
into a standard form that exhibits Y' as a cyclic cover of $\mathbb{A}^3$. To this
end let $B'' = B'^\omega = \mathbb{C}[x',y,\zeta,t']$ with a square root ω of the unity and
$\zeta = z'^2$. Put

$$x'' = -(x'^2 y + \zeta + t'^3 y).$$

Then $x' = x''y$ and we see that $B'' = \mathbb{C}[x'', y, t']$ is a polynomial ring in three variables and

$$x'' + x''^2 y^3 + \zeta + t'^3 y = 0.$$

Hence it follows that $B' = \mathbb{C}[x'', y, z', t']$ with a defining equation

$$x'' + x''^2 y^3 + z'^2 + t'^3 y = 0.$$

It now follows from [17, Theorem 8.4] that $\mathrm{ML}(Y) = \mathbb{C}[x]$ and $\mathrm{ML}(Y')$ $= B'$. $\qquad\square$

In order to show that a fixed point exists under a G_m-action on a $\mathbb{Q}$-homology threefold Y, we used the Smith theory and its variant. The following result without using the Smith theory is of some interest.

Lemma 4.6. *Let Y be a smooth affine variety with an effective G_m-action. Suppose that there are no fixed points. Then the Euler number $e(Y)$ of Y is zero.*

Proof. Let $q : Y \to X$ be the quotient morphism. Since there are no fixed points, every fiber is isomorphic to $\mathbb{A}^1_*$ when taken with reduced structure. The general fibers of q are reduced $\mathbb{A}^1_*$ and special fibers are multiple A^1_*. We work with the complex analytic topology. Considering the isotropy groups of the fibers, there exists a descending chain of closed subsets of X

$$F_0 \supset F_1 \supset \cdots \supset F_i \supset F_{i+1} \supset \cdots$$

such that $F_0 = X$ and the isotropy group of the fiber over a point of $F_i - F_{i+1}$ is constant, say G_i. Then $F_i - F_{i+1}$ is covered by open sets $\{U_{i\lambda}\}_{\lambda \in \Lambda_i}$ such that $q^{-1}(U_{i\lambda})_{\mathrm{red}} \cong G_m \times_{G_i} V_{i\lambda}$, where $V_{i\lambda}$ is a suitable slice (cf. [30, Théorème du slice étale et Remarque 3°]). Hence $q^{-1}(F_i)_{\mathrm{red}}$ is a $\mathbb{C}^*$-bundle over the open set $F_i - F_{i+1}$. This implies that the Euler number $e(Y)$ is zero. $\qquad\square$

Lemma 4.6 implies that any non-trivial G_m-action on a smooth affine variety Y has a fixed point if $e(Y) \neq 0$. Furthermore, considering the induced tangential representation at a fixed point, we know that the fixed point locus Y^{G_m} is smooth. However, we do not know if the fixed point locus Y^{G_m} is connected. The following example shows that the connectedness fails in general.

EXAMPLE **4.7.** Let $X = \mathbb{A}^2$ and let $Y_0 = X \times \mathbb{A}^1$ which has a standard G_m-action on the factor $\mathbb{A}^1$ with the point (0) as a fixed point. Choose two parallel lines ℓ_1, ℓ_2 on X, and let $Z_i = \ell_i \times \mathbb{A}^1$ for $i = 1, 2$. Let $\sigma : W \to Y_0$ be the blowing-ups with centers $\ell_1 \times (0)$ and $\ell_2 \times (0)$ and let Y be W with the proper transform of $X \times (0)$ removed. Then

the restriction of $p_1 \cdot \sigma$ onto Y gives a morphism $q : Y \to X$ which is eventually the quotient morphism by the G_m-action on Y induced by the one on Y_0, where p_1 is the projection $Y_0 \to X$. The singular fibers of q are crosses over $\ell_1 \cup \ell_2$. The locus of intersection points of crosses is the fixed point locus Y^{G_m}. Hence Y^{G_m} is not connected. The Euler number $e(Y)$ of the threefold Y is two. $\qquad\square$

In Section 2, we observed a G_m-action on a $\mathbb{Q}$-homology threefold Y such that the quotient morphism $q : Y \to X$ has relative dimension one. The following result deals with the case of q having relative dimension two.

Proposition 4.8. *Let Y be a $\mathbb{Q}$-homology threefold with a G_m-action. Suppose that the quotient morphism has equi-dimension two. Let $q : Y \to C$ be the quotient morphism, where C is a smooth affine curve. Then each fiber is isomorphic to $\mathbb{A}^2$, and C is isomorphic to $\mathbb{A}^1$. Hence Y is isomorphic to $\mathbb{A}^3$.*

Proof. Let F be a fiber of q. Since $\dim F = 2$, there is a unique fixed point Q such that the closure of every orbit passes through Q. The locus Γ of points Q is the fixed point locus Y^{G_m} and $q|_\Gamma : \Gamma \to C$ is a bijection. Hence it is an isomorphism. Since Y is smooth and the local intersection multiplicity $i(F, \Gamma; Q) = 1$, it follows that F is smooth near Q. Hence F itself is smooth. Then it is easy to show that F is isomorphic to $\mathbb{A}^2$ with an elliptic G_m-action. By a theorem of Sathaye [45], $q : Y \to C$ is an $\mathbb{A}^2$-bundle over C. Since Y is then contractable to C, the curve C is $\mathbb{Q}$-acyclic. Hence $C \cong \mathbb{A}^1$ and q is necessarily trivial. $\qquad\square$

Question 4.9. Let Y be a $\mathbb{Q}$-homology n-fold with a G_m-action. Suppose that the quotient morphism has equi-dimension $n-1$. Is Y isomorphic to the affine space $\mathbb{A}^n$?

In fact, one can show that each fiber of the quotient morphism $q : Y \to C$ is isomorphic to $\mathbb{A}^{n-1}$ with the induced elliptic G_m-action. If q is locally trivial in the Zariski topology, C is $\mathbb{Q}$-acyclic. Hence $C \cong \mathbb{A}^1$ and $Y \cong \mathbb{A}^n$. If Y is a homology n-fold, Question 4.9 has a positive answer which is a theorem of Kraft-Shwarz [29, Theorem 5].

REMARK **4.10.** In [25], a general result has been proved. Namely, let Y be a homology n-fold. Suppose that Y is dominated by an affine space and Y is endowed with an effective action of $T = G_m^{n-2}$ such that $\dim Y^T > 0$. Then Y is T-equivariantly isomorphic to the affine space $\mathbb{A}^n$ with a linear action of T. Hence a contractible threefold Y

having a non-hyperbolic G_m-action is G_m-equivariantly isomorphic to A^3 provided Y is dominated by an affine space.

The following result deals with more general cases of the quotient morphism having equi-dimension one.

Lemma 4.11. *Let Y be a smooth affine threefold with a G_m-action. Suppose that the quotient morphism $q : Y \to X$ has equi-dimension one. Then the following assertions hold.*

(1) *If $\dim Y^{G_m} = 2$, then Y^{G_m} is isomorphic to X and hence Y^{G_m} is connected. The morphism q defines a line bundle over X.*

(2) *If $\dim Y^{G_m} = 1$ and $e(Y) > 0$, then Y^{G_m} is smooth and consists of connected components $\Gamma_1, \ldots, \Gamma_r$, one of which is isomorphic to $\mathbb{A}^1$. Further, if Y^{G_m} is connected, then $e(Y) = 1$ and the quotient surface X is a normal affine surface with an embedded line and at worst cyclic quotient singularities.*

Proof.   (1) Since q does not contain a fiber component of dimension two, Y^{G_m} lies horizontally to the fibration q. Hence each fiber contains a unique fixed point. This implies that each fiber is isomorphic to $\mathbb{A}^1$. Considering the tangent space $T_{Y,Q}$ and the induced tangential representation for each $Q \in Y^{G_m}$, we know that Y^{G_m} is smooth and isomorphic to X. Namely, q is an $\mathbb{A}^1$-fibration with all reduced fibers isomorphic to $\mathbb{A}^1$ and has two cross-sections Y^{G_m} and a section at infinity. Hence q is in fact a line bundle.

(2) The morphism q is then an $\mathbb{A}^1_*$-fibration and Y^{G_m} is a smooth curve. Let $Y^{G_m} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_r$ be the decomposition into connected components. Let $\overline{\Gamma}_i$ be the smooth completion of Γ_i. Let g_i be the genus of $\overline{\Gamma}_i$ and let n_i be the number of points in $\overline{\Gamma}_i \setminus \Gamma_i$. Note that G_m acts on $q^{-1}(X - q(Y^{G_m}))$ without fixed points. Hence, by Lemma 4.6, the Euler number of $q^{-1}(X - q(Y^{G_m}))$ is zero. Note that $q^{-1}(q(\Gamma_i))$, taken with reduced structure, is a union of two $\mathbb{A}^1$-bundles over $q(\Gamma_i)$ meeting transversally along the section Γ_i. In fact, since the fiber $q^{-1}(P)$ over a point $P \in q(\Gamma_i)$ is a cross with each branch meeting Γ_i transversally in one point, $q^{-1}(q(\Gamma_i))$ consists of two irreducible components $W_i^{(1)}$ and $W_i^{(2)}$, each of which has an $\mathbb{A}^1$-fibration with a cross-section Γ_i. Hence $W_i^{(1)}$ and $W_2^{(2)}$ are $\mathbb{A}^1$ bundles over $q(\Gamma_i)$ meeting transversally along Γ_i. Hence the Euler number of $q^{-1}(q(\Gamma_i))$ is equal to $2 - 2g_i - n_i$. This observation yields a relation

$$0 < e(Y) = \sum_{i=1}^{r}(2 - 2g_i - n_i).$$

Since Γ_i is an affine curve, we have $n_i \geq 1$. If none of $\Gamma_1, \ldots, \Gamma_r$ is isomorphic to $\mathbb{A}^1$, then the right side of the above equality is less than or equal to zero, which is a contradiction. Hence one of them is isomorphic to $\mathbb{A}^1$. If Y^{G_m} is connected, then $Y^{G_m} = \Gamma_1 \cong \mathbb{A}^1$. It follows from the footnotes in the proof of Theorem 2.6 that q induces a closed embedding of Y^{G_m} into X and that X is smooth near $q(Y^{G_m})$. By Lemma 2.13, X has at worst cyclic quotient singularities in the open set $X \setminus (q(Y^{G_m}))$. $\qquad\square$

REMARK **4.12.** Given a smooth affine surface X_0, we can produce a smooth affine surface X by a half-point attachment [38, p. 233] which contains X_0 as an open set and has an embedded affine line C. By the same procedure as in Lemma 4.1, we take a product $X \times \mathbb{A}^1$ and apply the affine modification of $X \times \mathbb{A}^1$ with center $C \times (0)$ in $X \times (0)$. The resulting threefold is a smooth affine threefold Y with a G_m-action such that the quotient morphism $q : Y \to Y//G_m$, where $X \cong Y//G_m$ and $Y^{G_m} \cong C$. Further, $\overline{\kappa}(Y) = \overline{\kappa}(X)$.

References

[1] S.M. Bhatwadekar and A.K. Dutta, On $\mathbb{A}^*$-fibrations, J. Pure and Applied Algebra **149** (2000), 1– 14.

[2] A. Borel, Linear algebraic groups, Second edition, Graduate Texts in Mathematics **126**, Springer-Verlag, New York, 1991.

[3] A.K. Dutta, On $\mathbb{A}^1$-bundles of affine morphisms, J. Math. Kyoto Univ. **35** (3) (1995) 377–385.

[4] H. Flenner and M. Zaidenberg, Normal affine surfaces with $\mathbb{C}^*$-actions, Osaka J. Math. **40** (2003), no. 4, 981–1009.

[5] E.E. Floyd, On periodic maps and the Euler characteristics of associated spaces, Trans. Amer. Math. Soc. **72** (1952), 138–147.

[6] T. Fujita, On the topology of noncomplete algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), no. 3, 503–566.

[7] R.V. Gurjar, Two-dimensional quotients of $\mathbb{C}^n$ are isomorphic to $\mathbb{C}^2/\Gamma$, Transf. Groups, 12 (2007), no. 1, 117–125.

[8] R.V. Gurjar, M. Koras, M. Miyanishi and P. Russell, Affine normal surfaces with simply-connected smooth locus, Math. Ann. Published online: 06 July, 2011.

[9] R.V. Gurjar, K. Masuda and M. Miyanishi, $\mathbb{A}^1$-fibrations on affine threefolds, J. Pure and Applied Algebra, **216** (2012), 296–313.

[10] R.V. Gurjar and M. Miyanishi, Affine lines on logarithmic **Q**-homology planes, Math. Ann. 294 (1992), 463-482.

[11] R.V. Gurjar and M. Miyanishi, Pure subalgebras of polynomial rings, preprint.

[12] R.V. Gurjar and A.J. Parameswaran, Affine lines on $\mathbb{Q}$-homology planes, J. Math. Kyoto Univ. **35** (1995), no. 1, 63–77.

[13] H.A. Hamm, Zur Homotopietyp Steinscher Räume, J. Reine Angew. Math. **338** (1983), 121–135.

[14] Z. Jelonek, A number of points in the set $\mathbb{C}^2 \setminus F(\mathbb{C}^2)$, Bull. Polish Acad. Sci. Math. 47 (1999), no. 3, 257–261.

[15] Sh. Kaliman, Free $\mathbb{C}_+$-actions on $\mathbb{C}^3$ are translations, Invent. Math. **156** (2004), no. 1, 163–173.

[16] Sh. Kaliman, Actions of $\mathbb{C}^*$ and $\mathbb{C}_+$ on affine algebraic varieties, Algebraic geometry-Seattle 2005, Part 2, 629–654, Proc. Sympos. Pure Math., **80**, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[17] Sh. Kaliman and L. Makar-Limanov, On the Russell-Koras contractible threefolds, J. Algebraic Geom. **6** (1997), no. 2, 247–268.

[18] Sh. Kaliman and N. Saveliev, $\mathbb{C}_+$-actions on contractible threefolds, Michigan Math. J. **52** (2004), no. 3, 619–625.

[19] Sh. Kaliman and M. Zaidenberg, Affine modifications and affine hypersurfaces with a very transitive automorphism group, Transf. Groups **4** (1999), 53–95.

[20] Sh. Kaliman and M. Zaidenberg, Families of affine planes: the existence of a cylinder, Michigan Math. J. **49** (2001), no. 2, 353–367.

[21] T. Kambayashi and P. Russell, On linearizing algebraic torus actions, J. pure and applied algebra **23** (1982), 243–250.

[22] Y. Kawamata, On the classification of non-complete algebraic surfaces, Algebraic Geometry, Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, 215-232.

[23] Y. Kawamata, Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one, Proc. of the Internat. Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp. 207–217, Kinokuniya Book Store, Tokyo, 1978.

[24] M. Koras, Linearization of reductive group actions, in *Group Actions and Vector Fields*, Lecture Notes in Math. **956** (1982), 92–98.

[25] M. Koras and P. Russell, Codimension 2 torus actions on affine n-space, *Group actions and invariant theory* (Montreal, PQ, 1988), 103–110, CMS Conf. Proc. **10**, Amer. Math. Soc., Providence, RI, 1989.

[26] M. Koras, and P. Russell, Contractible threefolds and $\mathbb{C}^*$-actions on $\mathbb{C}^3$, J. Algebraic Geom. **6** (1997), no. 4, 671–695.

[27] H.P. Kraft, Geometrische Methoden in der Invariantentheorie, Aspects of Mathematics, D1. Friedr. Vieweg & Sohn, Braunschweig, 1984.

[28] H.P. Kraft, T. Petrie and J.D. Randall, Quotient varieties. Adv. Math. 74 (1989), no. 2, 145–162.

[29] H.P. Kraft and G.W. Schwarz, Reductive group actions with one-dimensional quotient, Inst. Hautes Etudes Sci. Publ. Math. No. **76** (1992), 1–97.

[30] D. Luna, Slices étales, Bull. Soc. Math. France, Paris, Memoire **33**, Soc. Math. France, Paris, 1973.

[31] K. Masuda and M. Miyanishi, Affine pseudo-planes and cancellation problem, Trans. Amer. Math. Soc. **357** (2005), 4867–4883.

[32] J. Milnor, Morse theory, Ann. Math. Study **51**, Princeton Univ. Press, 1963.

[33] M. Miyanishi, Normal affine subalgebras of a polynomial ring, Algebraic and Topological Theories - to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, 37-51.

[34] M. Miyanishi, An algebric characterization of the affine plane, J. Math. Kyoto Univ. **15**, (1975), 169-184.

[35] M. Miyanishi, An algebro-topological characterization of the affine space of dimension three, Amer. J. Math. **106** (1984), no. 6, 1469–1485.

[36] M. Miyanishi, Etale endomorphisms of algebraic varieties, Osaka J. Math. **22** (1985), 345–364.

[37] M. Miyanishi, Algebraic characterizations of the affine 3-space, Algebraic Geometry Seminar (Singapore, 1987), 53–67, World Sci. Publishing, Singapore, 1988.

[38] M. Miyanishi, Open algebraic surfaces, Centre de Recherches Mathématiques, Vol **12**, Université de Montréal, Amer. Math. Soc., 2001.

[39] M. Miyanishi, G_a-actions and completions, J. Algebra **319** (2008), 2845–2854.

[40] M. Miyanishi and T. Sugie, Homology planes with quotient singularities, J. Math. Kyoto Univ. **31** (1991), no. 3, 755–788.

[41] M.V. Nori, Zariski's conjecture and related problems. Ann. Sci. Ecole Norm. Sup. (4) 16 (1983), no. 2, 305–344.

[42] K. Palka, Singular $\mathbb{Q}$-homology planes I, arXiv:0806.3110.

[43] M. Rosenlicht, Some basic theorems on algebraic groups,. Amer. J. Math. **78** (1956), 401–443.

[44] P. Russell, Gradings of polynomial rings, Algebraic geometry and its applications (C.L. Bajaj ed.), Springer, 1994.

[45] A. Sathaye, Polynomial ring in two variables over a DVR: a criterion, Invent. Math. **74** (1983), no. 1, 159–168.

[46] E.H. Spanier, Algebraic topology, Springer Verlag, 1966.

[47] J. Winkelmann, A correspondence to P. Russell, August 23, 2011.

[48] M. Zaidenberg, Isotrivial families of curves on affine surfaces and characterization of the affine plane, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 534–567, 688; translation in Math. USSR-Izv. **30** (1988), no. 3, 503–532.

[49] O. Zariski, Interprétations algébrico-géométriques du quatorzième problème de Hilbert, Bull.Sci. Math. **78** (1954), 155-168.

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