DEFORMATIONS OF $\mathbb{A}^1$-FIBRATIONS

R.V. GURJAR, K. MASUDA AND M. MIYANISHI

Abstract. Let B be an integral domain which is finitely generated over a subdomain R and let D be an R-derivation on B such that the induced derivation $D_{\mathfrak{m}}$ on $B \otimes_R R/\mathfrak{m}$ is locally nilpotent for every maximal ideal $\mathfrak{m}$. We ask if D is locally nilpotent. Theorem 1.1 asserts that this is the case if B and R are affine domains. We next generalize the case of G_a-action treated in Theorem 1.1 to the case of $\mathbb{A}^1$-fibrations and consider the log deformations of affine surfaces with $\mathbb{A}^1$-fibrations. The case of $\mathbb{A}^1$-fibrations of affine type behaves nicely under log deformations, while the case of $\mathbb{A}^1$-fibrations of complete type is more involved (see Dubouloz-Kishimoto [3]). As a corollary, we prove the generic triviality of $\mathbb{A}^2$-fibration over a curve and generalize this result to the case of affine pseudo-planes.

Introduction

An $\mathbb{A}^1$-fibration $\rho : X \to B$ on a smooth affine surface X to a smooth curve B is given as the quotient morphism of a G_a-action if the parameter curve B is an affine curve (see [8]). Meanwhile, it is not so if B is a complete curve. When we deform the surface X under a suitable setting (log deformation), our question is if the neighboring surfaces still have $\mathbb{A}^1$-fibrations of affine type or of complete type according to the type of the $\mathbb{A}^1$-fibration on X being affine or complete. Assuming that the neighboring surfaces have $\mathbb{A}^1$-fibrations, the propagation of the property of $\mathbb{A}^1$-fibration being of affine type or of complete type is proved in Lemma 2.2, whose proof reflects the structure of the boundary divisor at infinity of an affine surface with $\mathbb{A}^1$-fibration. The stability of the boundary divisor under small deformations, e.g., the stability of the weighted dual graphs has been discussed in topological methods (e.g., [25]). Furthermore, if such property is inherited by the neighboring

Date: June 22, 2013.

2000 *Mathematics Subject Classification.* Primary: 14R20; Secondary: 14R25.

Key words and phrases. $\mathbb{A}^1$-fibration, locally nilpotent derivation, affine threefold, deformation, generic triviality, affine pseudo-plane.

The second and third authors are supported by Grant-in-Aid for Scientific Research (C), No. 22540059 and (B), No. 24340006, JSPS.

surfaces, we still ask if the ambient threefold has an $\mathbb{A}^1$-fibration or equivalently if the generic fiber has an $\mathbb{A}^1$-fibration.

The answer to this question is subtle. We consider first in the section one the case where each of the fiber surfaces of the deformation has an $\mathbb{A}^1$-fibration of affine type induced by a global vector field on the ambient threefold. This global vector field is in fact given by a locally nilpotent derivation (Theorem 1.1). If the $\mathbb{A}^1$-fibrations on the fiber surfaces are of affine type, we can show (Theorem 2.6) that there exists an $\mathbb{A}^1$-fibration on the ambient threefold such that the $\mathbb{A}^1$-fibration on each general fiber surface is induced by the global one up to an automorphism of the fiber surface. The proof of Theorem 2.6 depends on Lemma 2.2 which we prove by observing the behavior of the boundary rational curves. This is done by the use of Hilbert scheme (see [20]).

As a consequence, we can prove the generic triviality of an $\mathbb{A}^2$-fibration over a curve. Namely, if $f : Y \to T$ is a smooth morphism from a smooth affine threefold to a smooth affine curve such that the fiber over every *closed* point of T is isomorphic to the affine plane $\mathbb{A}^2$, then the generic fiber of f is isomorphic to $\mathbb{A}^2$ over the function field $k(T)$ of T and f is an $\mathbb{A}^2$-bundle over an open set of T (see Theorem 2.8). This fact, together with a theorem of Sathaye [28], shows that f is an $\mathbb{A}^2$-bundle over T in the Zariski topology.

The question on the generic triviality is also related to a question on the triviality of a k-form of a surface with an $\mathbb{A}^1$-fibration (see Conjecture 2.11). In the case of an $\mathbb{A}^1$-fibration of complete type, the answer is negative by Dubouloz-Kishimoto [3] (see Theorem 5.1).

Theorem 2.8 was proved by our predecesors Kaliman-Zaidenberg [15] in a more comprehensive way and without assuming that the base is a curve. The idea in our first proof of Theorem 2.8 is of more algebraic nature and consists of using the existence of a locally nilpotent derivation on the coordinate ring of Y and the second proof of using the Ramanujam-Morrow graph of the normal minimal completion of $\mathbb{A}^2$ was already used in [15]. The related results are also discussed in the article [27].

The algebro-geometric arguments using Hilbert scheme in the section two can be replaced by topological arguments using Ehresmann's theorem which might be more appreciated than the use of Hilbert scheme. This is done in the section three.

In the section four, we extend the above result on the generic triviality of an $\mathbb{A}^2$-fibration over a curve by replacing $\mathbb{A}^2$ by an affine pseudo-plane which has properties similar to $\mathbb{A}^2$, e.g., the boundary divisor for a minimal normal completion is a linear chain of rational curves. An affine pseudo-plane is a $\mathbb{Q}$-homology plane, and we note that Flenner-Zaidenberg [5] made a fairly exhaustive consideration for the log deformations of $\mathbb{Q}$-homology planes.

In the final section five, we observe the case of $\mathbb{A}^1$-fibration of complete type and show by an example of Dubouloz-Kishimoto [3] that the ambient threefold does not have an $\mathbb{A}^1$-fibration. But it is still plausible that the ambient threefold is affine uniruled in the stronger sense that the base change of the ambient deformation space by a suitable lifting of the base curve has a global $\mathbb{A}^1$-fibration. But this still remains open.

As a final remark, we note that a preprint of Flenner-Kaliman-Zaidenberg [6] recently uploaded on the web treats also deformations of surfaces with $\mathbb{A}^1$-fibrations.

1. Triviality of deformations of locally nilpotent derivations

Let k be an algebraically closed field of characteristic zero which we fix as the ground field. Let $Y = \mathrm{Spec} B$ be an irreducible affine algebraic variety. We define the tangent sheaf $\mathcal{T}_{Y/k}$ as $\mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_{Y/k}, \mathcal{O}_Y)$. A regular vector field on Y is an element of $\Gamma(Y, \mathcal{T}_{Y/k})$. A regular vector field Θ on Y is identified with a derivation D on B via isomorphisms

$$\Gamma(Y, \mathcal{T}_{Y/k}) \cong \mathrm{Hom}_B(\Omega^1_{B/k}, B) \cong \mathrm{Der}_k(B, B).$$

We say that Θ is *locally nilpotent* if so is D. In the first place, we are interested in finding a necessary and sufficient condition for D to be locally nilpotent. Suppose that Y has a fibration $f : Y \to T$. A natural question is to ask whether D is locally nilpotent if the restriction of D on each closed fiber of f is locally nilpotent. The following result shows that this is the case[1].

Theorem 1.1. *Let $Y = \mathrm{Spec}\, B$ and $T = \mathrm{Spec}\, R$ be irreducible affine varieties defined over k and let $f : Y \to T$ be a dominant morphism such that general fibers are irreducible and reduced. We consider R to be a subalgebra of B. Let D be an R-trivial derivation of B such that, for each closed point $t \in T$, the restriction $D_t = D \otimes_R R/\mathfrak{m}$ is a locally nilpotent derivation of $B \otimes_R R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R corresponding to t. Then D is locally nilpotent.*

We need some preliminary results. We retain the notations and assumptions in the above theorem.

Lemma 1.2. *There exist a finitely generated field extension k_0 of the prime field $\mathbb{Q}$ which is a subfield of the ground field k, geometrically integral affine varieties $Y_0 = \mathrm{Spec}\, B_0$ and $T_0 = \mathrm{Spec}\, R_0$, a dominant*

[1]The result is also remarked in [3, Remark 13].

morphism $f_0 : Y_0 \to T_0$ and an R_0-trivial derivation D_0 of B_0 such that the following conditions are satisfied:

(1) *Y_0, T_0, f_0 and D_0 are defined over k_0.*
(2) *$Y = Y_0 \otimes_{k_0} k, T = T_0 \otimes_{k_0} k, f = f_0 \otimes_{k_0} k$ and $D = D_0 \otimes_{k_0} k$.*
(3) *D_0 is locally nilpotent if and only if so is D.*

Proof. Since B and R are integral domains finitely generated over k, write B and R as the residue rings of certain polynomial rings over k modulo the finitely generated ideals. Write $B = k[x_1, \ldots, x_r]/I$ and $R = k[t_1, \ldots, t_s]/J$. Furthermore, the morphism f is determined by the images $f^*(\eta_j) = \varphi_j(\xi_1, \ldots, \xi_r)$ in B, where $\xi_i = x_i \pmod I$ and $\eta_j = t_j \pmod J$. Adjoin to $\mathbb{Q}$ all coefficients of the finite generators of I and J as well as the coefficients of the φ_j to obtain a subfield k_0 of k. Let $B_0 = k_0[x_1, \ldots, x_r]/I_0$ and $R_0 = k_0[t_1, \ldots, t_s]/J_0$, where I_0 and J_0 are respectively the ideals in $k_0[x_1, \ldots, x_r]$ and $k_0[t_1, \ldots, t_s]$ generated by the same generators of I and J. Furthermore, define the homomorphism f_0^* by the assignment $f_0^*(\eta_j) = \varphi_j(\xi_1, \ldots, \xi_r)$. Let $Y_0 = \operatorname{Spec} B_0$, $T_0 = \operatorname{Spec} R_0$ and let $f_0 : Y_0 \to T_0$ be the morphism defined by f_0^*. The derivation D corresponds to a B-module homomorphism $\delta : \Omega^1_{B/R} \to B$. Since $\Omega^1_{B/R} = \Omega^1_{B_0/R_0} \otimes_{k_0} k$, we can enlarge k_0 so that there exists a B_0-homomorphism $\delta_0 : \Omega^1_{B_0/R_0} \to B_0$ satisfying $\delta = \delta_0 \otimes_{k_0} k$. Let $D_0 = \delta_0 \cdot d_0$, where $d_0 : B_0 \to \Omega^1_{B_0/R_0}$ is the standard differentiation. Then we have $D = D_0 \otimes_{k_0} k$.

Let $\Phi_0 : B_0 \to B_0[[u]]$ be the R_0-homomorphism into the formal power series ring in t over B_0 defined by

$$\Phi_0(b_0) = \sum_{i \geq 0} \frac{1}{i!} D_0^i(b_0) u^i \ .$$

Let $\Phi : B \to B[[u]]$ be the R-homomorphism defined in a similar fashion. Then Φ_0 and Φ are determined by the images of the generators of B_0 and B. Since the generators of B_0 and B are the same, we have $\Phi = \Phi_0 \otimes_{k_0} k$. The derivation D_0 is locally nilpotent if and only if Φ_0 splits via the polynomial subring $B_0[u]$ of $B_0[[u]]$. This is the case for D as well. Since Φ_0 splits via $B_0[u]$ if and only if Φ splits via $B[u]$, D_0 is locally nilpotent if and only if so is D. $\square$

Lemma 1.3. *Let k_1 be the algebraic closure of k_0 in k. Let $Y_1 = \operatorname{Spec} B_1$ with $B_1 = B_0 \otimes_{k_0} k_1$, $T_1 = \operatorname{Spec} R_1$ with $R_1 = R_0 \otimes_{k_0} k_1$ and $f_1 = f_0 \otimes_{k_0} k_1$. Let $D_1 = D_0 \otimes_{k_0} k_1$. Then the following assertions hold.*

(1) *Let t_1 be a closed point of T_1. Then the restriction of D_1 on the fiber $f_1^{-1}(t_1)$ is locally nilpotent.*
(2) *D_1 is locally nilpotent if and only if so is D.*

Proof. (1) Let t be the unique closed point of T lying over t_1 by the projection morphism $T \to T_1$, where $R = R_1 \otimes_{k_1} k$. (If $\mathfrak{m}_1$ is the maximal ideal of R_1 corresponding to t_1, $\mathfrak{m}_1 \otimes_{k_1} k$ is the maximal ideal of R corresponding to t.) Then $F_t = f^{-1}(t) = f_1^{-1}(t_1) \otimes_{k_1} k$ and the restriction D_t of D onto F_t is given as $D_{1,t_1} \otimes_{k_1} k$, where D_{1,t_1} is the restriction of D_1 onto $f_1^{-1}(t_1)$. We consider also the R-homomorphism $\Phi : B \to B[[u]]$ and the R_1-homomorphism $\Phi_1 : B_1 \to B_1[[u]]$. As above, let $\mathfrak{m}$ and $\mathfrak{m}_1$ be the maximal ideals of R and R_1 corresponding to t and t_1. Then D_t gives rise to the $R/\mathfrak{m}$-homomorphism $\Phi \otimes_R R/\mathfrak{m} : B \otimes_R R/\mathfrak{m} \to (B \otimes_R R/\mathfrak{m})[[u]]$. Similarly, D_{1,t_1} gives rise to the $R_1/\mathfrak{m}_1$-homomorphism $\Phi_1 \otimes_{R_1} R_1/\mathfrak{m}_1 : B_1 \otimes_{R_1} R_1/\mathfrak{m}_1 \to (B_1 \otimes_{R_1} R_1/\mathfrak{m}_1)[[u]]$, where $R/\mathfrak{m} = k$ and $R_1/\mathfrak{m}_1 = k_1$. Then $\Phi \otimes_R R/\mathfrak{m} = (\Phi_1 \otimes_{R_1} R_1/\mathfrak{m}_1) \otimes_{k_1} k$. Hence $\Phi \otimes_R R/\mathfrak{m}$ splits via $(B \otimes_R R/\mathfrak{m})[u]$ if and only if $\Phi_1 \otimes_{R_1} R_1/\mathfrak{m}_1$ splits via $(B_1 \otimes_{R_1} R_1/\mathfrak{m}_1)[u]$. Hence $D_{1,t}$ is locally nilpotent as so is D_t.

(2) The same argument as above using the homomorphism Φ can be applied. □

The field k_0 can be embedded into the complex field $\mathbb{C}$ because it is a finitely generated field extension of $\mathbb{Q}$. Hence we can extend the embedding $k_0 \hookrightarrow \mathbb{C}$ to the algebraic closure k_1. Thus k_1 is viewed as a subfield of $\mathbb{C}$. Then Lemma 1.3 holds if one replaces the extension k/k_1 by the extension $\mathbb{C}/k_1$. Hence it suffices to prove Theorem 1.1 with an additional hypothesis $k = \mathbb{C}$.

Lemma 1.4. *Theorem 1.1 holds if k is the complex field $\mathbb{C}$.*

Proof. Let $Y(\mathbb{C})$ be the set of closed points which we view as a complex analytic space embedded into a complex affine space $\mathbb{C}^N$ as a closed set. Consider the Euclidean metric on $\mathbb{C}^N$ and the induced metric topology on $Y(\mathbb{C})$. Then $Y(\mathbb{C})$ is a complete metric space.

Let b be a nonzero element of B. For a positive integer m, define a Zariski closed subset $Y_m(b)$ of $Y(\mathbb{C})$ by

$$Y_m(b) = \{Q \in Y(\mathbb{C}) \mid D^m(b)(Q) = 0\} .$$

Since Q lies over a closed point t of $T(\mathbb{C})$ and D_t is locally nilpotent on $f^{-1}(t)$ by the hypothesis, we have

$$f^{-1}(t) \subset \bigcup_{m>0} Y_m(b) .$$

This implies that $Y(\mathbb{C}) = \bigcup_{m>0} Y_m(b)$. We claim that $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$. In fact, this follows by Baire category theorem, which states that if the $Y_m(b)$ are all proper closed subsets, its countable union cannot cover the uncountable set $Y(\mathbb{C})$. If $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$ then $D^m(b) = 0$. This implies that D is locally nilpotent on B.

One can avoid the use of Baire category theorem in the following way. Suppose that $Y_m(b)$ is a proper closed subset for every $m > 0$. Let H be a general hyperplane in $\mathbb{C}^N$ such that the section $Y(\mathbb{C}) \cap H$ is irreducible, $\dim Y(\mathbb{C}) \cap H = \dim Y(\mathbb{C}) - 1$, and $Y(\mathbb{C}) \cap H = \bigcup_{m>0} (Y_m(b) \cap H)$ with $Y_m(b) \cap H$ a proper closed subset of $Y(\mathbb{C}) \cap H$ for every $m > 0$. We can further take hyperplane sections and find a general linear subspace L in $\mathbb{C}^N$ such that $Y(\mathbb{C}) \cap L$ is an irreducible curve and $Y(\mathbb{C}) \cap L = \bigcup_{m>0} (Y_m(b) \cap L)$, where $Y_m(b) \cap L$ is a proper Zariski closed subset. Hence $Y_m(b) \cap L$ is a finite set, and $\bigcup_{m>0} (Y_m(b) \cap L)$ is a countable set, while $Y(\mathbb{C}) \cap L$ is not a countable set. This is a contradiction. Thus $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$. □

Let D be a k-derivation on a k-algebra B. It is called *surjective* if D is so as a k-linear mapping. The follwoing result is a consequence of Theorem 1.1

Corollary 1.5. *Let $Y = \text{Spec}\, B$, $T = \text{Spec}\, R$ and $f : Y \to T$ be the same as in Theorem 1.1. Let D be an R-derivation of B such that D_t is a surjective k-derivation for every closed point $t \in T$. Assume further that the relative dimension of f is one. Then D is a locally nilpotent derivation and f is an $\mathbb{A}^1$-fibration.*

Proof. Let t be a closed point of T such that the fiber $f^{-1}(t)$ is irreducible and reduced. By [9, Theorem 1.2 and Proposition 1.7], the coordinate ring $B \otimes R/\mathfrak{m}$ of $f^{-1}(t)$ is a polynomial ring $k[x]$ in one variable and $D_t = \partial/\partial x$, where $\mathfrak{m}$ is the maximal ideal of R corresponding to t. Then D_t is locally nilpotent. Taking the base change of $f : Y \to T$ by $U \hookrightarrow T$ if necessary, where U is a small open set of T, we may assume that D_t is locally nilpotent for every closed point t of T. By Theorem 1.1, the derivation D is locally nilpotent and hence f is an $\mathbb{A}^1$-fibration. □

2. Deformations of $\mathbb{A}^1$-fibrations of affine type

In the present section, we assume that the ground field k is the complex field. Let X be an affine algebraic surface which is normal. Let $p : X \to C$ be an $\mathbb{A}^1$-fibration, where C is an algebraic curve which is either affine or projective. We say that the $\mathbb{A}^1$-fibration p is of *affine type* (resp. *complete type*) if C is affine (resp. complete). The $\mathbb{A}^1$-fibration on X is the quotient morphism of a G_a-action on X if and only if it is of affine type (see [8]). We consider the following result on deformations. For the complex analytic case, one can refer to [18] and also to [12, p. 269].

Lemma 2.1. *Let $\overline{f} : \overline{Y} \to T$ be a smooth projective morphism from a smooth algebraic threefold $\overline{Y}$ to a smooth algebraic curve. Let C be a smooth rational complete curve contained in $\overline{Y}_0 = \overline{f}^{-1}(t_0)$ for a closed point t_0 of T [2]. Then the following assertions hold.*

(1) *The Hilbert scheme $\mathrm{Hilb}(\overline{Y})$ has dimension less than or equal to $h^0(C, \mathrm{N}_{C/\overline{Y}})$ in the point $[C]$. If $h^1(C, \mathrm{N}_{C/\overline{Y}}) = 0$ then the equality holds and $\mathrm{Hilb}(\overline{Y})$ is smooth at $[C]$. Here $\mathrm{N}_{C/\overline{Y}}$ denotes the normal bundle of C in $\overline{Y}$.*

(2) *Let $n = (C^2)$ on $\overline{Y}_0$. Then $\mathrm{N}_{C/\overline{Y}} \cong \mathcal{O}_C \oplus \mathcal{O}_C(n)$ provided $n \geq -1$.*

(3) *Suppose $n = 0$. Then, with T replaced by its Zariski open set if necessary, the morphism $\overline{f}$ splits as*

$$\overline{f} : \overline{Y} \xrightarrow{\varphi} V \xrightarrow{\sigma} T ,$$

where φ is a $\mathbb{P}^1$-fibration with C contained as a fiber and σ makes V a T-scheme of relative dimension one.

(4) *Suppose $n = -1$. Then C does not deform in the fiber $\overline{Y}_0$ but deforms along the morphism $\overline{f}$ after an étale finite base change. Namely, there are an étale finite morphism $\sigma : T' \to T$ and an irreducible subvariety Z of codimension one in $\overline{Y}' := \overline{Y} \times_T T'$ such that Z can be contracted along the fibers of $\overline{f}' : \overline{Y}' \to T'$, where Y' is an irreducible smooth affine curve and $\overline{f}'$ is the second projection of $Y \times_T T'$ to T'.*

(5) *Assume that there are no (-1)-curves E and E' in $\overline{Y}_0$ such that $E \cap E' \neq \emptyset$ and E is algebraically equivalent to E' as 1-cycles on $\overline{Y}$. Then, after shrinking T to a smaller open set if necessary, we can take Z in the assertion (4) above as a subvariety of $\overline{Y}$. The contraction of Z gives a factorization $\overline{f}|_Z : Z \xrightarrow{g} T' \xrightarrow{\sigma} T$, where g is a $\mathbb{P}^1$-fibration, C is a fiber of g and σ is as above.*

Proof. (1) The assertion follows from Grothendieck [7, Cor. 5.4].

(2) We have an exact sequence

$$0 \to \mathrm{N}_{C/\overline{Y}_0} \to \mathrm{N}_{C/\overline{Y}} \to \mathrm{N}_{\overline{Y}_0/\overline{Y}}|_C \to 0 ,$$

[2]When we write $t \in T$, we tacitly assume that t is a closed point of T

where $\mathrm{N}_{C/\overline{Y}_0} \cong \mathcal{O}_C(n)$ and $\mathrm{N}_{\overline{Y}_0/\overline{Y}}|_C \cong \mathcal{O}_C$. The obstruction for this exact sequence to split lies in $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(n)) \cong \mathrm{H}^1(C, \mathcal{O}_C(n))$, which is zero if $n \geq -1$.

(3) Suppose $n = 0$. Then $\dim_{[C]} \mathrm{Hilb}(\overline{Y}) = 2$ and $[C]$ is a smooth point of $\mathrm{Hilb}(\overline{Y})$. Let H be a relatively ample divisor on $\overline{Y}/T$ and set $P(n) := P_C(n) = h^0(C, \mathcal{O}_C(nH))$ the Hilbert polynomial in n of C with respect to H. Then $\mathrm{Hilb}^P(\overline{Y})$ is a scheme which is projective over T. Let V be the irreducible component of $\mathrm{Hilb}^P(\overline{Y})$ containing the point $[C]$. Then V is a T-scheme with a morphism $\sigma : V \to T$, $\dim V = 2$ and V has relative dimension one over T. Furthermore, there exists a subvariety Z of $\overline{Y} \times_T V$ such that the fibers of the composite morphism

$$g : Z \hookrightarrow \overline{Y} \times_T V \xrightarrow{p_2} V$$

are curves on $\overline{Y}$ parametrized by V. For a general point $v \in V$, the corresponding curve $C' := C_v$ is a smooth rational complete curve because $P_{C'}(n) = P(n)$ and $(C')^2 = 0$ on $\overline{Y}_t = \overline{f}^{-1}(t)$ with $t = \sigma(v)$ because $\dim_{[C']} \mathrm{Hilb}(\overline{Y}) = 2$. In fact, if $(C')^2 \leq -1$ then the exact sequence of normal bundles in (2) implies $h^0(C', \mathrm{N}_{C'/\overline{Y}}) \leq 1$, which contradicts $\dim_{[C']} \mathrm{Hilb}(\overline{Y}) = 2$. If $(C')^2 > 0$ then $\dim_{[C']} \mathrm{Hilb}(\overline{Y}) > 2$, which is again a contradiction. So, $(C')^2 = 0$. Hence $\overline{Y}_t$ has a $\mathbb{P}^1$-fibration $\varphi_t : \overline{Y}_t \to \overline{B}_t$ such that C' is a fiber, where $\overline{B}_t$ is a smooth complete curve. Considering the correspondence of the curves, we have a birational mapping from a projective curve $V_t := \sigma^{-1}(t) \to \overline{B}_t$ which turns out to be an isomorphism. Thus there exists a Zariski open set T' of T such that $(\sigma \cdot g)^{-1}(T')$ is isomorphic to $\overline{f}^{-1}(T')$ as T'-schemes.

(4) Suppose $n = -1$. Then $h^0(C, \mathrm{N}_{C/\overline{Y}}) = 1$ and $h^1(C, \mathrm{N}_{C/\overline{Y}}) = 0$. Hence $\mathrm{Hilb}^P(\overline{Y})$ has dimension one and is smooth at $[C]$, where $P(n) = P_C(n)$ is the Hilbert polynomial of C with respect to H. Let T' be the irreducible component of $\mathrm{Hilb}^P(\overline{Y})$ containing $[C]$. Note that $\dim T' = 1$. Then we find a subvariety Z in $\overline{Y} \times_T T'$ such that C is a fiber of g and every fiber of the T-morphism $g = p_2|_{T'} : Z \to T'$ is a (-1) curve in the fiber $\overline{Y}_t$. In fact, the nearby fibers of C are (-1) curves as a small deformation of C by [18]. Hence, by covering T' by small disks, we know that every fiber of g is a (-1) curve. Further, the projection $\sigma : T' \to T$ is a finite morphism as it is projective and T' is smooth because each fiber is a (-1) curve in $\overline{Y}$ (see the above arugument for $[C]$). Furthermore, σ is étale since $\overline{f}$ is locally a product of the fiber and the base in the Euclidean topology. Hence σ induces a local isomorphism between T' and T. This implies that $\overline{Y} \times_T T'$ is a smooth affine threefold and the second projection $\overline{f}' : \overline{Y} \times_T T' \to T'$ is

a smooth projective morphism. Now, after an étale finite base change $\sigma : T' \to T$, we may assume that Z is identified with a subvariety of $\overline{Y}$. Since C is a (-1) curve in $\overline{Y}_0$, it is an extremal ray in the cone $\overline{NE}(\overline{Y}_0)$. Since C is algebraically equivalent to the fibers of $g : Z \to T'$, it follows that C is an extremal ray in the relative cone $\overline{NE}(\overline{Y}/T)$. Then it follows from [21, Theorem 3.25] that Z is contracted along the fibers of g in $\overline{Y}$ and the threefold obtained by the contraction is smooth and projective over T.

(5) Let $\sigma^{-1}(t_0) = \{u_1, \ldots, u_d\}$ and let $Z_{u_i} = Z \cdot (\overline{Y} \times \{u_i\})$ for $1 \le i \le d$. Then the Z_{u_i} are the (-1) curves on $\overline{Y}_0$ which are algebraically equivalent to each other as 1-cycles on $\overline{Y}$. By the assumption, $Z_{u_i} \cap Z_{u_j} = \emptyset$ whenever $i \neq j$. This property holds for all $t \in T$ if one shrinks to a smaller open set of t_0. Then we can identify Z with a closed subvariety of $\overline{Y}$. In fact, the projection $p : Z \hookrightarrow Y \times_Y T' \to Y$ is a T-morphism. For the point $t_0 \in T$, the morphism $p \otimes_{\mathcal{O}_{T,t_0}} \widehat{\mathcal{O}}_{T,t_0}$ with the completion $\widehat{\mathcal{O}}_{T,t_0}$ of $\mathcal{O}_{T,t_0}$ is a direst sum of the closed immersions from $Z \otimes_{\mathcal{O}_{T,t_0}} \widehat{\mathcal{O}}_{T',u_i}$ into $Y \otimes_{\mathcal{O}_{T,t_0}} \widehat{\mathcal{O}}_{T',u_i}$ for $1 \le i \le r$. So, $p \otimes_{\mathcal{O}_{T,t_0}} \widehat{\mathcal{O}}_{T,t_0}$ is a closed immersion. Hence p is a closed immersion locally over t_0 because $\widehat{\mathcal{O}}_{T,t_0}$ is faithfully flat over $\mathcal{O}_{T,t_0}$. The rest is the same as in the proof of the assertion (4). □

Let Y_0 be a smooth affine surface and let $\overline{Y}_0$ be a smooth projective surface containing Y_0 as an open set in such a way that the complement $\overline{Y}_0 \backslash Y_0$ supports a reduced effective divisor D_0 with simple normal crossings. We call $\overline{Y}_0$ a *normal completion* of Y_0 and D_0 the boundary divisor of Y_0. An irreducible component of D_0 is called a (-1) *component* if it is a smooth rational curve with self-intersection number -1. We say that $\overline{Y}_0$ is a *minimal* normal completion if the contraction of an (-1) component of D_0 (if any) results the image of D_0 losing the condition of simple normal crossings.

Let $\overline{f} : \overline{Y} \to T$ be a smooth projective morphism from a smooth algebraic threefold $\overline{Y}$ to a smooth algebraic curve T and let $S = \sum_{i=1}^r S_i$ be a reduced effective divisor on $\overline{Y}$ with simple normal crossings. Let $Y = \overline{Y} \setminus S$ and let $f = \overline{f}|_Y$. We assume that for every point $t \in T$, the intersection cycle $D_t = \overline{f}^{-1}(t) \cdot S$ is a reduced effective divisor of $\overline{Y}_t = \overline{f}^{-1}(t)$ with simple normal crossings and $Y_t = Y \cap \overline{Y}_t$ is an affine open set of $\overline{Y}_t$. For a point $t_0 \in T$, we assume that $\overline{Y}_{t_0} = \overline{Y}_0, D_{t_0} = D_0$ and $Y_{t_0} = Y_0$. A collection $(Y, \overline{Y}, S, \overline{f}, t_0)$ is called a *family of logarithmic deformations* of a triple $(Y_0, \overline{Y}_0, D_0)$. We call it simply a *log deformation* of the triple $(Y_0, \overline{Y}_0, D_0)$. Since f is smooth and S is a divisor with simple normal crossings, $(Y, \overline{Y}, S, \overline{f}, t_0)$ is a family of logarithmic deformations in the sense of Kawamata [16, 17].

From time to time, we have to make a base change by an étale finite morphism $\sigma : T' \to T$ with irreducible T'. Let $\overline{Y}' = \overline{Y} \times_T T', \overline{f}' = \overline{f} \times_T T', S' = S \times_T T'$ and $Y' = Y \times_T T'$. Since the field extension $k(\overline{Y})/k(T)$ is a regular extension, $\overline{Y}'$ is an irreducible smooth projective threefold, and S' is a divisor with simple normal crossings. Hence $(Y', \overline{Y}', S', \overline{f}', t'_0)$ is a family of logarithmic deformations of the triple $(Y'_0, \overline{Y}'_0, D'_0) \cong (Y_0, \overline{Y}_0, D_0)$, where $t'_0 \in T'$ with $\sigma(t'_0) = t_0$.

We have the following result on logarithmic deformations of affine surfaces with $\mathbb{A}^1$-fibrations.

Lemma 2.2. *Let $(Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of the triple $(Y_0, \overline{Y}_0, D_0)$. Then the following assertions hold.*

(1) *Assume that Y_0 has an $\mathbb{A}^1$-fibration. Then Y_t has an $\mathbb{A}^1$-fibration for every $t \in T$.*
(2) *If Y_0 has an $\mathbb{A}^1$-fibration of affine type (resp. of complete type), then Y_t has also an $\mathbb{A}^1$-fibration of affine type (resp. of complete type) for every $t \in T$.*

Proof. (1) Note that $K_{\overline{Y}_t} = (K_{\overline{Y}} + \overline{Y}_t) \cdot \overline{Y}_t = K_{\overline{Y}} \cdot \overline{Y}_t$ because $\overline{Y}_t$ is algebraically equivalent to $\overline{Y}_{t'}$ for $t' \neq t$. Then $K_{\overline{Y}_t} + D_t = (K_{\overline{Y}} + S) \cdot \overline{Y}_t$. By the hypothesis, $h^0(\overline{Y}_0, \mathcal{O}(n(K_{\overline{Y}_0} + D_0))) = 0$ for every $n > 0$. Then the semicontinuity theorem [11, Theorem 12.8] implies that $h^0(\overline{Y}_t, \mathcal{O}(n(K_{\overline{Y}_t} + D_t))) = 0$ for every $n > 0$. Hence $\overline{\kappa}(Y_t) = -\infty$. Since Y_t is affine, this implies that Y_t has an $\mathbb{A}^1$-fibration.

(2) Suppose that Y_0 has an $\mathbb{A}^1$-fibration $\rho_0 : Y_0 \to B_0$ which is of affine type. Then ρ_0 defines a pencil Λ_0 on $\overline{Y}_0$.

Suppose first that Λ_0 has no base points and hence defines a $\mathbb{P}^1$-fibration $\overline{\rho}_0 : \overline{Y}_0 \to \overline{B}_0$ such that $\overline{\rho}_0|_{Y_0} = \rho_0$ and $\overline{B}_0$ is a smooth completion of B_0. If $\overline{\rho}_0$ is not minimal, let E be a (-1) curve contained in a fiber of $\overline{\rho}_0$, which is necessarily not contained in Y_0. By Lemma 2.1, E extends along the morphism $\overline{f}$ if one replaces the base T by a suitable étale finite covering T' and can be contracted simultaneously with other (-1) curves contained in the fibers $\overline{Y}_t$ $(t \in T)$. Note that this étale finite change of the base curve does not affect the properties of the fiber surfaces. Hence we may assume that all simultaneous blowing-ups and contractions as applied below are achieved over the base T.

The contraction is performed either within the boundary divisor S or the *simultaneous half-point detachments* in the respective fibers Y_t

for $t \in T$. (For the definition of half-point detachment (resp. attachment), see for example [4]). Hence the contraction does not change the hypothesis on the simple normal crossing of S and the intersection divisor $S \cdot \overline{Y}_t$. Thus we may assume that $\overline{\rho}_0$ is minimal. Since $B_0 \subsetneqq \overline{B}_0$, a fiber of $\overline{\rho}_0$ is contained in a boundary component, say S_1. Then the intersection $S_1 \cdot \overline{Y}_0$ as a cycle is a disjoint sum of the fibers of $\overline{\rho}_0$ with multiplicity one. Hence $(S_1^2 \cdot \overline{Y}_0) = ((S_1 \cdot \overline{Y}_0)^2)_{\overline{Y}_0} = 0$. Since $\overline{Y}_t$ and $\overline{Y}_0$ are algebraically equivalent, we have $(S_1^2 \cdot \overline{Y}_t) = 0$ for every $t \in T$. Note that $\overline{Y}_t$ is also a ruled surface by Iitaka [12] and minimal by the same reason as for $\overline{Y}_0$. Considering the deformations of a fiber of $\overline{\rho}_0$ appearing in $S_1 \cdot \overline{Y}_0$, we know by Lemma 2.1 that $S_1 \cdot \overline{Y}_t$ is a disjoint sum of smooth rational curves with self-intersection number zero. Namely, $S_1 \cdot \overline{Y}_t$ is a sum of the fibers of a $\mathbb{P}^1$-fibration. Here we may have to replace the $\mathbb{P}^1$-fibration $\overline{\rho}_t$ by the second one if $\overline{Y}_t \cong \mathbb{P}^1 \times \mathbb{P}^1$. In fact, if a smooth complete surface has two different $\mathbb{P}^1$-fibrations and is minimal with respect to one fibration, then the surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and two $\mathbb{P}^1$-fibrations are the vertical and horizontal fibrations. This implies that Y_t has an $\mathbb{A}^1$-fibration of affine type.

Suppose next that Λ_0 has a base point, say P_0, and that the $\mathbb{A}^1$-fibration ρ_0 is of affine type. Then all irreducible components of $D_0 := S \cdot \overline{Y}_0$ are contained in the members of Λ_0. Since the boundary divisor D_0 of $\overline{Y}_0$ is assumed to be a connected divisor with simple normal crossings, there are at most two components of $S \cdot \overline{Y}_0$ passing through P_0, and if there are two of them, they lie on different components of S and P_0 lies on their intersection curve. In particular, if S_1 is a component of S containing P_0, then $S_1 \cdot \overline{Y}_0$ is a disjoint sum of smooth rational curves. Let C_1 be the component of $S_1 \cdot \overline{Y}_0$ passing through P_0 and let F_0 be the member of Λ_0 which contains C_1. We may assume that F_0 is supported by the boundary divisor D_0. If F_0 contains a (-1) curve E such that $P_0 \notin E$, then E extends along the morphism $\overline{f}$ and can be contracted simultaneously along $\overline{f}$. So, we may assume that every irreducible component of F_0 not passing P_0 has self-intersection number ≤ -2 on $\overline{Y}_0$. Then we may assume that $(C_1^2)_{\overline{Y}_0} \geq 0$. In fact, if there are two irreducible components of $S \cdot \overline{Y}_0$ passing through P_0 and belonging to the same member F_0 of Λ_0, one of them must have self-intersection number ≥ 0, for otherwise all the components of the member of Λ_0, after the elimination of base points, would have self-intersection number ≤ -2, which is a contradiction. So, we may assume that the one on S_1, i.e., C_1, has self-intersection number ≥ 0. Then the proper transform of C_1 is the unique (-1) curve with multiplicity > 1

in the fiber corresponding to F_0 after the elimination of base points of Λ_0.

On the other hand, $S_1 \cdot \overline{Y}_0$ (as well as $S_i \cdot \overline{Y}_0$ if it is non-empty) is a disjoint sum of smooth rational curves, one of which is C_1. If there is another curve in $S_1 \cdot \overline{Y}_0$ other than C_1, then it is connected to the component of D_0 different from C_1 and passing through the point P_0. This is impossible because Y_0 has an $\mathbb{A}^1$-fibration and some irreducible component of S has to "cross" this $\mathbb{A}^1$-fibration. So, $S_1 \cdot \overline{Y}_0 = C_1$. Let $n := (C_1^2)_{\overline{Y}_0} \geq 0$. Then $\mathrm{Hilb}^P(\overline{Y})$ has dimension $n+2$ and is smooth at the point $[C_1]$. Since $C_1 \cong \mathbb{P}^1$ and $\mathrm{N}_{C_1/\overline{Y}} \cong \mathcal{O}(n) \oplus \mathcal{O}$, C_1 extends along the morphism $\overline{f}$. Namely, $\overline{f}|_{S_1} : S_1 \to T$ is a composite of a $\mathbb{P}^1$-fibration $\sigma_1 : S_1 \to T'$ and an étale finite morphism $\sigma_2 : T' \to T$, where C_1 is a fiber of σ_1. Since $S_1 \cdot \overline{Y}_0 = C_1$, the morphism σ_2 is the identity morphism. So, $\sigma_1 = \overline{f}|_{S_1}$ and $C_1 = \sigma_1^{-1}(t_0)$. In particular, $(C_1^2)_{S_1} = 0$.

Suppose that C_2 is a component of F_0 meeting C_1. Then C_2 is contained in a different boundary component, say S_2, which intersects S_1. Since $(H \cdot S_2 \cdot \overline{Y}_0) > 0$, we have $(H \cdot S_2 \cdot \overline{Y}_t) > 0$ for every $t \in T$, where H is a relatively ample divisor on $\overline{Y}$ over T. Furthermore, $S_2 \cdot \overline{Y}_0$ is algebraically equivalent to $S_2 \cdot \overline{Y}_t$. Note that $S_2 \cdot \overline{Y}_0$ is a disjoint sum of smooth rational curves, one of which is connected to C_1. By the same reason as for $S_1 \cdot \overline{Y}_0$, it follows that $S_2 \cdot \overline{Y}_0 = C_2$. Hence we have

$$(S_1 \cdot S_2 \cdot \overline{Y}_t) = (S_1 \cdot S_2 \cdot \overline{Y}_0) = (C_1 \cdot (S_2 \cdot \overline{Y}_0))_{\overline{Y}_0} = (C_1 \cdot C_2)_{\overline{Y}_0} = 1.$$

This implies that $S_2 \cdot \overline{Y}_t$ is irreducible for a general point $t \in T$. For otherwise, by the Stein factorization of the morphism $\overline{f}|_{S_2} : S_2 \to T$, the fiber $S_2 \cdot \overline{Y}_t$ is a disjoint sum $A_1 + \cdots + A_s$ of distinct irreducible curves which are algebraically equivalent to each other on S_2. Since

$$\begin{aligned} 1 = (S_1 \cdot S_2 \cdot \overline{Y}_t) &= ((S_1 \cdot S_2) \cdot (S_2 \cdot \overline{Y}_t))_{S_2} \\ &= ((S_1 \cdot S_2) \cdot (A_1 + \cdots + A_s))_{S_2} = s((S_1 \cdot S_2) \cdot A_1), \end{aligned}$$

we have $s = 1$ and $(S_1 \cdot S_2 \cdot A_1) = 1$. So, $\sigma_2 := \overline{f}|_{S_2} : S_2 \to T$ is a $\mathbb{P}^1$-bundle and $(C_2^2)_{S_2} = 0$. This implies that $\mathrm{N}_{C_2/\overline{Y}} \cong \mathcal{O}(m) \oplus \mathcal{O}$ with $m = (C_2^2)_{\overline{Y}_0} \leq -2$ and that C_2 extends along the morphism $\overline{f}$. We can argue in the same way as above with irreducible components of F_0 other than C_1.

Assume that no members of Λ_0 except F_0 have irreducible components outside of Y_0. If C_i is shown to move on the component S_i along the morphism $\overline{f}$, we consider a component C_{i+1} anew which meets C_i. Each of them is contained in a distinct irreducible boundary component of S and extends along the morphism $\overline{f}$. Let $S_1, S_2, \ldots, S_r$ be all the boundary components which meet $\overline{Y}_0$ along the irreducible components of F_0. Then $\overline{Y}_t$ intersects $S_1 + S_2 + \cdots + S_r$ in an effective divisor which has the same form as F_0. Furthermore, we have

$$((S_i \cdot \overline{Y}_t)^2)_{\overline{Y}_t} = (S_i^2 \cdot \overline{Y}_t) = (S_i^2 \cdot \overline{Y}_0) = ((S_i \cdot \overline{Y}_0)^2)_{\overline{Y}_0}$$

for $1 \le i \le r$. Namely, the components $S_i \cdot \overline{Y}_t$ $(1 \le i \le r)$ with the same multiplicities as $S_i \cdot \overline{Y}_0$ in F_0 is a member F_t of the pencil Λ_t lying outside of Y_t. This implies that Λ_t has a base point P_t and at least one member of Λ_t lies outside of Y_t. So, the $\mathbb{A}^1$-fibration ρ_t on Y_t is of affine type.

If the pencil Λ_0 contains two members F_0, F_0' such that the components C_1, C_1' of F_0, F_0' lie outside of Y_0 and pass through the point P_0, we may assume that F_0 is supported by the boundary components, while F_0' may not. Then no other members of Λ_0 have irreducible components outside of Y_0 because $\overline{Y}_0 \setminus Y_0$ is connected. We can argue as above to show that the member F_0 moves along the morphism $\overline{f}$, and further that every boundary component of F_0' moves on a boundary component, say S_j', as a fiber of $f|_{S_j'} : S_j' \to T$. Hence the pencil Λ_t has the member F_t corresponding to F_0 whose all components lie outside of Y_t and the member F_t' corresponding to F_0'. In fact, the part of F_t' lying outside of Y_t is determined as above, but since $Y_t \cap F_t'$ is a disjoint union of the $\mathbb{A}^1$ which correspond to the (-1) components of F_t' (the *half-point attachments*), the member F_t' is determined up to its weighted graph. This proof also implies that if ρ_0 is of complete type then ρ_t is of complete type for every $t \in T$. □

Concerning the possibility of achieving the contractions over the base curve T, we have the following result.

Lemma 2.3. *Let $(Y, \overline{Y}, S, \overline{f}, t_0)$ be a family of logarithmic deformation of the triple $(Y_0, \overline{Y}_0, D_0)$. Assume that Y_0 has an $\mathbb{A}^1$-fibration of affine type. Let Λ_0 be the pencil on $\overline{Y}_0$ whose general members are the closures of fibers of the $\mathbb{A}^1$-fibration. Then there are no two (-1) curves E_1 and E_2 such that they belong to the same connected component of the Hilbert scheme $\mathrm{Hilb}^P(\overline{Y})$, E_1 is an irreducible component of a member of Λ_0 and $E_1 \cap E_2 \neq \emptyset$.*

Proof. Suppose that such E_1 and E_2 exist. Since E_1 and E_2 are algebraically equivalent 1-cycles on $\overline{Y}$, E_1 and E_2 have the same intersections with subvarieties of codimension one in $\overline{Y}$, in particular, with the boundary divisor D_0 including the infinitesimal components which arise from the simultaneous blowing-ups along $\overline{f}$.

Consider first the case where Λ_0 has no base points and hence Λ_0 induces a $\mathbb{P}^1$-fibration. If both E_1 and E_2 are contained in the fiber at infinity, i.e., the one supported by D_0, then $E_1 \cap E_2 = \emptyset$ for otherwise the fiber at infinity contains a loop. If E_1 only is contained in the fiber at infinity, E_2 is also a fiber component because E_1 (hence E_2) does not intersect a general fiber of Λ_0 which has a small deformation in $\overline{Y}$ and hence $E_1 \cap E_2 = \emptyset$. If E_1 and E_2 are not contained in the fiber at infinity, E_1 and E_2 are the fiber components of different fibers of Λ_0 because they intersect a component of D_0 in the same fashion, and hence $E_1 \cap E_2 = \emptyset$.

Even if Λ_0 has a base point P_0, $E_1 \cap E_2 = \emptyset$ follows unless both E_1 and E_2 pass through the point P_0 and belong to different members of Λ_0, say F_1 and F_2. After the elimination of base points of Λ_0, the intersection numbers of the proper transforms of E_1 and E_2 are less than -1. Hence there are (-1) components in the respective members F_1 and F_2 which lie inside Y_0. This is a contradiction because Y_0 is affine. □

Remark 2.4. In the step of the above proof of Lemma 2.2 where we assume that no members of Λ_0 except F_0 have irreducible components outside of Y_0, let P'_t be a point on $C_{1,t} := S_1 \cdot \overline{Y}_t$ other than P_t which is the base point of the given pencil Λ_t. Then there is a pencil Λ'_t on $\overline{Y}_t$ which is similar to Λ_t. In fact, note first that $\overline{Y}_t$ is a rational surface. Perform the same blowing-ups with centers at P'_t and its infinitely near points as those with centers at P_t and its infinitely near points which eliminate the base points of Λ_t. Then we find an effective divisor $\widetilde{F}'_t$ supported by the proper transforms of $S_i \cdot \overline{Y}_t$ $(1 \le i \le r)$ and the exceptional curves of the blowing-ups such that $\widetilde{F}'_t$ has the same form and multiplicities as the corresponding member $\widetilde{F}_t$ in the proper transform $\widetilde{\Lambda}_t$ of Λ_t after the elimination of base points. Then $(\widetilde{F}'_t)^2 = 0$ and hence $\widetilde{F}'_t$ is a fiber of an $\mathbb{P}^1$-fibration on the blown-up surface of $\overline{Y}_t$. Then the fibers of the $\mathbb{P}^1$-fibration form the pencil Λ'_t on $\overline{Y}_t$ after the reversed contractions. In fact, the surface $Y_t = \overline{Y}_t \setminus D_t$ is the affine plane with two systems of coordinate lines given as the fibers of Λ_t and Λ'_t. Hence the $\mathbb{A}^1$-fibrations induced by Λ_t and Λ'_t are transformed by an automorphism of $\overline{Y}_t$. □

The following is one of the simplest examples of our situation.

Example 2.5. Let C be a smooth conic and let S be the subvariety of codimension one in $\mathbb{P}^2 \times C$ defined by

$$S = \{(P, Q) \mid P \in L_Q,\ Q \in C\},$$

where L_Q is the tangent line of C at Q. Let $Y = (\mathbb{P}^2 \times C) \setminus S$ and let $f : Y \to C$ be the projection onto C. We set $T = C$ to fit the previous notations. Set $\overline{Y} = \mathbb{P}^2 \times C$. Then $\overline{f} : \overline{Y} \to T$ is the second projection and the boudary divisor S is irreducible. For every point $Q \in C$, $Y_Q := \mathbb{P}^2 \setminus L_Q$ has a linear pencil Λ_Q generated by C and $2L_Q$, which induces an $\mathbb{A}^1$-fibration of affine type. The restriction $\overline{f}|_S : S \to T$ is a $\mathbb{P}^1$-bundle. Let C be defined by $X_0X_2 = X_1^2$ with respect to a system of homogeneous coordinates (X_0, X_1, X_2) of $\mathbb{P}^2$ and let $\eta = (1, t, t^2)$ be the generic point of C with t an inhomogeneous coordinate on $C \cong \mathbb{P}^1$. Then L_η is defined by $t^2X_0 - 2tX_1 + X_2 = 0$. The generic fiber Y_η of f has an $\mathbb{A}^1$-fibration induced by the linear pencil Λ_η whose general members are the conics defined by $(X_0X_2-X_1^2)+u(t^2X_0-2tX_1+X_2)^2 = 0$, where $u \in \mathbb{A}^1$. Indeed, the conics are isomorphic to $\mathbb{P}^1_{k(t)}$ since they have the $k(t)$-rational point $(1, t, t^2)$, and Y_η is isomorphic to $\mathbb{A}^2_{k(t)}$. This implies that the affine threefold Y itself has an $\mathbb{A}^1$-fibration. □

We prove one of our main theorems.

Theorem 2.6. *Let $f : Y \to T$ be a morphism from a smooth affine threefold onto a smooth curve T with irreducible general fibers. Assume that general fibers of f have $\mathbb{A}^1$-fibrations of affine type. Then, after shrinking T if necessary, Y has an $\mathbb{A}^1$-fibration which factors f.*

Proof. Embed Y into a smooth threefold $\overline{Y}$ in such a way that f extends to a projective morphism $\overline{f} : \overline{Y} \to T$. We may assume that the complement $S := \overline{Y} \setminus Y$ is a reduced divisor with simple normal crossings. Let $S = S_1 + S_2 + \cdots + S_r$ be the irreducible decomposition of S. For a general point $t \in T$, let Y_t be the fiber $f^{-1}(t)$ and let $\rho_t : Y_t \to B_t$ be the given $\mathbb{A}^1$-fibration on Y_t. By the assumption, B_t is an affine curve. We may assume that Y_t is smooth and hence B_t is smooth. Let $\overline{Y}_t$ be the closure of Y_t in $\overline{Y}$ which we may assume to be a smooth projective surface with t a general point of T. By replacing T by a smaller Zariski open set, we may assume that $\overline{f}$ is a smooth morphism and that $S \cdot \overline{Y}_t$ is a divisor with simple normal crossings for every $t \in T$. We have two cases to consider.

Case 1. The fibration ρ_t extends to a $\mathbb{P}^1$-fibration $\overline{\rho}_t : \overline{Y}_t \to \overline{B}_t$ for general points $t \in T$, where $\overline{B}_t$ is a smooth completion of B_t. For $t_0 \in T$, we consider the fibration $\overline{\rho}_0 : \overline{Y}_0 \to \overline{B}_0$. A general fiber of $\overline{\rho}_0$ meets one of the irreducible components S_i, say S_1, in one point. Then so does every fiber of $\overline{\rho}_0$ because $S_1 \cdot \overline{Y}_0$ is a divisor on $\overline{Y}_0$ and the fibers of $\overline{\rho}_0$ are algebraically equivalent to each other. We claim that

(1) $\overline{Y}_t$ meets the component S_1 for every $t \in T$.

(2) After possibly switching the $\mathbb{A}^1$-fibrations if some $\overline{Y}_t$ has two $\mathbb{A}^1$-fibrations, we may assume that for every $t \in T$, the fibers of the $\mathbb{P}^1$-fibration $\overline{\rho}_t$ on $\overline{Y}_t$ meet S_1 along a curve $\overline{A}_t$ contained in S_1 such that $\overline{A}_t$ is a cross-section of $\overline{\rho}_t$ and hence $\overline{\rho}_t$ induces an isomorphism between $\overline{A}_t$ and $\overline{B}_t$.

In fact, for a relatively ample divisor H of $\overline{Y}$ over T, we have $(H \cdot S_1 \cdot \overline{Y}_0) > 0$, whence $(H \cdot S_1 \cdot \overline{Y}_t) > 0$ for every $t \in T$ because $\overline{Y}_t$ is algebraically equivalent to $\overline{Y}_0$. This implies the assertion (1). To prove the assertion (2), we consider the deformation of a smooth fiber C of $\overline{\rho}_0$ in $\overline{Y}_0$. By Lemma 2.1, there is a $\mathbb{P}^1$-fibration $\varphi : \overline{Y} \to V$ such that C is a fiber of φ. Then the restriction $\varphi_0 = \varphi|_{\overline{Y}_0}$ is the $\mathbb{P}^1$-fibration $\overline{\rho}_0$. For every $t \in T$, the restriction $\varphi_t = \varphi|_{\overline{Y}_t}$ is a $\mathbb{P}^1$-fibration on $\overline{Y}_t$. If φ_t is different from $\overline{\rho}_t$, we replace $\overline{\rho}_t$ by φ_t. Then $(S_1 \cdot C') = (S_1 \cdot C) = 1$ for a general fiber C' of φ_t because C' is algebraically equivalent to C. The assertion follows immediately.

With the notations in the proof of Lemma 2.1, the isomorphisms $\overline{A}_t \xrightarrow{\sim} V_t := \sigma^{-1}(t) \cong \overline{B}_t$ shows that the morphism

$$S_1 \hookrightarrow \overline{Y} \xrightarrow{\varphi} V \xrightarrow{\sigma} T$$

induces a birational T-morphism $S_1 \to V$ and S_1 is a cross-section of φ. It is clear that the boundary divisor S contains no other components which are horizontal to φ. Hence Y has an $\mathbb{A}^1$-fibration.

Case 2. Suppose that ρ_t does not extend to a $\mathbb{P}^1$-fibration for a general $t \in T$. Then the closures of the fibers of ρ_t form a linear pencil Λ_t on $\overline{Y}_t$ with a base point, say P_t. Since B_t is an affine curve, we are now in the situation (the case of $\mathbb{A}^1$-fibrations of affine type) treated in the assertion (2) of Lemma 2.2 and its proof. We take a cross-section T_1 of the $\mathbb{P}^1$-bundle $\sigma_1 : S_1 \to T$ such that $P_0 \in T_1$. Here we note that a $\mathbb{P}^1$-bundle over a smooth curve has a cross-section by Tsen's theorem. If P_0 is not on the intersection curve of the irreducible components of S, we shrink T so that T_1 does not meet any intersection curve of S. Let $P'_t = T_1 \cap \overline{Y}_t$. Then P'_t may differ from the base point P_t of the pencil Λ_t. We then replace Λ_t by the pencil Λ'_t which is the isomorphic image of Λ_t (see the previous remark). Now we blow up $\overline{Y}$ with center at T_1 and replace $\overline{Y}$ by the new threefold $\overline{Y}'$ which contains the given affine threefold Y as an open set. Thus we can alleviate the base condition of the linear pencil Λ_t. After the first blowing-up, the proper transform of Λ_t on $\overline{Y}'$ has the base point (if any) lying on the intersection of the proper transform of S_1 and the exceptional divisor, and hence there is no need to replace the linear pencil by a similar pencil. Repeating this process for finitely many times we are reduced to the case 1. Hence Y has an $\mathbb{A}^1$-fibration. □

As a consequence of Theorem 2.6, we have the following result.

Corollary 2.7. *Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold Y to a smooth affine curve T. Assume that f has the relative projective completion $\overline{f} : \overline{Y} \to T$ which satisfies the same conditions on the boundary divisor S and the intersection of each fiber $\overline{Y}_t$ with S as set in Lemma 2.2. If a fiber Y_0 has a G_a-action, then the threefold Y has a G_a-action as a T-scheme.*

Proof. By Lemma 2.2, every fiber Y_t has an $\mathbb{A}^1$-fibration of affine type $\rho_t : Y_t \to B_t$, where B_t is an affine curve. By Theorem 2.6, Y has an $\mathbb{A}^1$-fibration $\rho : Y \to U$ such that f is factored as

$$f : Y \xrightarrow{\rho} U \xrightarrow{\sigma} T \ ,$$

where $U_t := \sigma^{-1}(t) \cong B_t$ for every $t \in T$. Then U is an affine scheme after restricting T to a Zariski open set. Then Y has a G_a-action by [8]. □

Given a smooth affine morphism $f : Y \to T$ from a smooth algebraic variety Y to a smooth curve T such that every *closed* fiber is isomorphic to the affine space $\mathbb{A}^n$ of fixed dimension, one can ask if the generic fiber of f is isomorphic to $\mathbb{A}^n$ over the function field $k(T)$. If this is the case with f, we say that *the generic triviality* holds for f. In the case $n = 2$, this holds by the following theorem. If the generic triviality for $n = 2$ holds for $f : Y \to T$ in the setup of Theorem 2.8, a theorem of Sathaye [28] shows that f is an $\mathbb{A}^2$-bundle in the sense of Zariski topology.

Theorem 2.8. *Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold Y to a smooth affine curve T. Assume that the fiber Y_t is isomorphic to $\mathbb{A}^2$ for every closed point of T. Then the generic fiber Y_η of f is isomorphic to the affine plane over the function field of T. Hence $f : Y \to T$ is an $\mathbb{A}^2$-bundle over T after replacing T by an open set if necessary.*

Before giving a proof, we prepare a lemma where an integral k-scheme is a reduced and irreducible algebraic k-scheme.

Lemma 2.9. *Let $p : X \to T$ be a dominant morphism from an integral k-scheme X to an integral k-scheme T. Assume that the fiber X_t is an integral k-scheme for every closed point t of T. Then the generic fiber $X_\eta = X \times_T \operatorname{Spec} k(T)$ is geometrically integral $k(T)$-scheme.*

Proof. We have only to show that the extension of the function fields $k(X)/k(T)$ is a regular extension. Namely, $k(X)/k(T)$ is a separable extension, i.e., a separable algebraic extension of a transcendental extension of $k(T)$ and $k(T)$ is algebraically closed in $k(X)$. Since the characteristic of k is zero, it suffices to show that $k(T)$ is algebraically closed in $k(X)$. Suppose the contrary. Let K be the algebraic closure of $k(T)$ in $k(X)$, which is a finite algebraic extension of $k(T)$. Let T' be the normalization of T in K. Let $\nu : T' \to T$ be the normalization morphism which is a finite morphism. Then $p : X \to T$ splits as $p : X \xrightarrow{p'} T' \xrightarrow{\nu} T$, which is the Stein factorization. Then the fiber X_t is not irreducible for a general closed point $t \in T$, which is a contradiction to the hypothesis. □

Proof of Theorem 2.8. Every closed fiber Y_t has an $\mathbb{A}^1$-fibration of affine type. By Theorem 2.6, Y has an $\mathbb{A}^1$-fibration which induces $\mathbb{A}^1$-fibrations on general closed fibers Y_t. The $\mathbb{A}^1$-fibration on Y is induced by a G_a-fibration [8] which is induced by a locally nilpotent derivation δ on the coordinate ring B of Y, i.e., $Y = \operatorname{Spec} B$. Let $T = \operatorname{Spec} R$. Here δ is an R-trivial derivation on B. Let A be the kernel of δ. Since B is a smooth k-algebra of dimension 3, A is a finitely generated, normal k-algebra of dimension 2. The derivation δ induces a locally nilpotent derivation δ_t on $B_t = B \otimes_R R/\mathfrak{m}_t$, where $\mathfrak{m}_t$ is the maximal ideal of R corresponding to a general point t of T. We assume that $\delta_t \neq 0$. Since B_t is a polynomial k-algebra of dimension 2 by the hypothesis, $A_t := \operatorname{Ker} \delta_t$ is a polynomial ring of dimension 1.

Claim 1. *$A_t = A \otimes_R R/\mathfrak{m}_t$ if δ_t is nonzero.*

Proof. Let $\varphi : B \to B[u]$ be the k-algebra homomorphism defined by

$$\varphi(b) = \sum_{i \geq 0} \frac{1}{i!} \delta^i(b) u^i \ .$$

Then $\operatorname{Ker} \delta = \operatorname{Ker} (\varphi - \mathrm{id}\ )$. Hence we have an exact sequence of R-modules

$$0 \to A \to B \xrightarrow{\varphi - \mathrm{id}} B[u] \ .$$

Let $\mathcal{O}_t$ be the local ring of T at t, i.e., the localization of R with respect to $\mathfrak{m}_t$, and let $\widehat{\mathcal{O}}_t$ be the $\mathfrak{m}_t$-adic completion of $\mathcal{O}_t$. Since $\widehat{\mathcal{O}}_t$ is a flat R-module, we have an exact sequence

$$0 \to A \otimes_R \widehat{\mathcal{O}}_t \to B \otimes_R \widehat{\mathcal{O}}_t \to (B \otimes_R \widehat{\mathcal{O}}_t)[u] \ . \qquad (*)$$

The completion $\widehat{\mathcal{O}}_t$ as a k-module decomposes as $\widehat{\mathcal{O}}_t = k \oplus \widehat{\mathfrak{m}}_t$, where $\widehat{\mathfrak{m}}_t = \mathfrak{m}_t \widehat{\mathcal{O}}_t$, the above exact sequence splits as a direct sum of exact sequences of k-modules

$$0 \to A \otimes_R k \to B \otimes_R k \to (B \otimes_R k)[u]$$
$$0 \to A \otimes_R \widehat{\mathfrak{m}}_t \to B \otimes_R \widehat{\mathfrak{m}}_t \to (B \otimes_R \widehat{\mathfrak{m}}_t)[u] \ .$$

The first one is, in fact, equal to

$$0 \to A \otimes_R R/\mathfrak{m}_t \to B_t \xrightarrow{\varphi_t - \mathrm{id}} B_t[u] \ ,$$

where φ_t is defined by δ_t in the same way as φ by δ. Hence $\mathrm{Ker}\ \delta_t = A \otimes_R R/\mathfrak{m}_t = A_t$. □

Let $X = \mathrm{Spec}\, A$ and let $p : X \to T$ be the morphism induced by the inclusion $R \hookrightarrow A$. Thus $f : Y \to T$ splits as

$$f : Y \xrightarrow{q} X \xrightarrow{p} T \ ,$$

where q is the quotient morphism by the induced G_a-action on Y.

Claim 2. *Suppose that $\delta_t \neq 0$ for every $t \in T$. Then X is a smooth surface with $\mathbb{A}^1$-bundle structure over T.*

Proof. Note that R is a Dedekind domain and A is an integral domain. Hence p is a flat morphism. Since f is surjective, p is also surjective. Hence p is a faithfully flat morphism. Further, by Claim 1, $X_t = \mathrm{Spec}\ (A \otimes_R R/\mathfrak{m}_t)$ is equal to $\mathrm{Spec}\ A_t$ for every t, which is isomorphic to $\mathbb{A}^1$. In fact, the kernel of a non-trivial locally nilpotent derivation on a polynomial ring of dimension 2 is a polynomial ring of dimension 1. The generic fiber of p is geometrically integral by Lemma 2.9. Hence, by [13, Theorem 2], X is an $\mathbb{A}^1$-bundle over T. In particular, X is smooth. □

Let $K = k(T)$ be the function field of T. The generic fiber $X_K = X \times_T \mathrm{Spec}\, K$ is geometrically integral as shown in the above proof of Claim 2.

Claim 3. *The generic fiber $Y_K = Y \times_T \mathrm{Spec}\, K$ is isomorphic to $\mathbb{A}^2_K$.*

Proof. We consider $q_K : Y_K \to X_K$, where $X_K \cong \mathbb{A}^1_K$. We prove the following two assertions.

(1) For every closed point x of X_K, the fiber $Y_K \times_{X_K} \mathrm{Spec}\, K(x)$ is isomorphic to $\mathbb{A}^1_{K(x)}$.
(2) The generic fiber of q_K is geometrically integral.

Note that $K(x)$ is a finite algebraic extension of K. Let T' be the normalization of T in $K' := K(x)$. We consider $Y' := Y \times_T T'$ instead of Y. Then the G_a-action on Y lifts to Y' and the quotient variety is $X' = X \times_T T'$. Indeed, the normalization R' of R in K' is the coordinate ring of T' and is a flat R-module. Then the sequence of R'-modules

$$0 \to A \otimes_R R' \to B \otimes_R R' \xrightarrow{\varphi' - \mathrm{id}} (B \otimes_R R')[u]$$

is exact, where $\varphi' = \varphi \otimes_R R'$. Hence $q_{K'} : Y'_{K'} \to X'_{K'}$, which is the base change of q_K with respect to the field extension K'/K, is the quotient morphism by the G_a-action on $Y'_{K'}$ induced by δ. Since $X'_{K'} = X \times_T \operatorname{Spec} K'$, there exists a K'-rational point x' on $X'_{K'}$ such that x is the image of x' by the projection $X'_{K'} \to X_K$. If the fiber of $q_{K'}$ over x', i.e., $Y'_{K'} \times_{X'_{K'}} (\operatorname{Spec} K', x')$, is isomorphic to $\mathbb{A}^1_{K'}$, then $Y_K \times_{X_K} \operatorname{Spec} K'$ is isomorphic to $\mathbb{A}^1_{K'}$ because $Y'_{K'} \times_{X'_{K'}} \operatorname{Spec} K' = Y_K \times_{X_K} \operatorname{Spec} K'$. Thus we may assume that x is a K-rational point. Let C be the closure of x in X. Then C is a cross-section of $p : X \to T$. Let $Z := Y \times_X C$. Then $q_C : Z \to C$ is a faithfully flat morphism such that the fiber $q_C^{-1}(w)$ is isomorphic to $\mathbb{A}^1$ for every closed point $w \in C$. In fact, $q_C^{-1}(w)$ is the fiber of $Y_t \to X_t$ over the point $w \in C$, where $t = p(w), Y_t \cong \mathbb{A}^2, X_t \cong \mathbb{A}^1$ and $X_t = Y_t/\!/G_a$. By Lemma 2.9 (which is extended to a non-closed field K), the generic fiber of q_C is geometrically integral, and the generic fiber of q_C, which is $Y_K \times_{X_K} \operatorname{Spec} K(x)$, is isomorphic to $\mathbb{A}^1_{K'}$ by [13, Theorem 2]. This proves the first assertion.

The generic point of X_K corresponds to the quotient field $L := Q(A)$. Then it suffices to show that $B \otimes_A Q(A)$ is geometrically integral over $Q(A)$. Meanwhile, $B \otimes_A Q(A)$ has a locally nilpotent derivation $\delta \otimes_A Q(A)$ such that $\operatorname{Ker}(\delta \otimes_A Q(A)) = Q(A)$. Hence $B \otimes_A Q(A)$ is a polynomial ring $Q(A)[u]$ in one variable over $Q(A)$ because $\delta \otimes_A Q(A)$ has a slice. So, $B \otimes_A Q(A)$ is geometrically integral over $Q(A)$. Now, by [14, Theorem], Y_K is an $\mathbb{A}^1$-bundle over $X_K \cong \mathbb{A}^1_K$. Hence Y_K is isomorphic to $\mathbb{A}^2_K$. We have to replace T by an open set $T \setminus F$, where $F = \{t \in T \mid \delta_t = 0\}$. This completes the proof of Theorem 2.8. □

We can prove Theorem 2.8 in a more geometric way by making use of a theorem of Ramanujam-Morrow on the boundary divisor of a minimal normal completion of the affine plane [26, 24]. The proof given below is explained in more precise and explicit terms in [15, Lemma 3.2]. In particular, the step to show that $\overline{Y}_K \cong \mathbb{P}^2_K$ and $Y_K \cong \mathbb{A}^2_K$ is due to [*loc.cit.*].

The second proof of Theorem 2.8. Let $f : Y \to T$ be as in Theorem 2.8. Let $\overline{Y}$ be a relative completion such that $\overline{Y}$ is smooth and f extends to a smooth projective morphism $\overline{f} : \overline{Y} \to T$ such that the conditions in Lemma 2.2 are satisfied together with $S := \overline{Y} \setminus Y$. To obtain this setting, we may have to shrink T to a smaller open set of

T. In particular, we assume that $\overline{Y}_t$ is a smooth normal completion of Y_t for every closed point $t \in T$, where Y_t is isomorphic to $\mathbb{A}^2$. Fix one such completed fiber, say $\overline{Y}_0 = \overline{f}^{-1}(t_0)$, and consider the reduced effective divisor $\overline{Y}_0 \setminus Y_0$ with $Y_0 = f^{-1}(t_0) \cong \mathbb{A}^2$. Namely, $(Y, \overline{Y}, S, \overline{f}, t_0)$ is a log deformation of $(\overline{Y}_0, D_0, Y_0)$. If the dual graph of this divisor is not linear then it contains a (-1)-curve meeting at most two other components of D_0 by a result of Ramanujam [26]. By (4) of Lemma 2.1, such a (-1)-curve deforms along the fibers of $\overline{f}$ and we get an irreducible component, say S_1, of $S = \sum_{i=0}^{r} S_i$ which can be contracted. Repeating this argument, we can assume that all the dual graphs for $\overline{Y}_t \setminus Y_t$, as t varies on the set of closed points of T, are linear chains of smooth rational curves. By [24], at least one of these curves is a (0)-curve. Fix such a (0)-curve C_1 in $\overline{Y}_0 \setminus Y_0$. Then C_1 deforms along the fibers of $\overline{f}$ and forms an irreducible component, say S_1, of S by abuse of the notations. By the argument in the proof of Lemma 2.2, if C_2 is a component of $\overline{Y}_0 \setminus Y_0$ meeting C_1, it deforms along the fibers of $\overline{f}$ on an irreducible component, say S_2, of S. Repeating this argument, we know that all irreducible components of $\overline{Y}_0 \setminus Y_0$ extend along the fibers of $\overline{f}$ to form the irreducible components of S and that the dual graphs of $\overline{Y}_t \setminus Y_t$ are the same for every $t \in T$. Now let K be the function field of T over k. We consider the generic fibers $\overline{Y}_K$ and Y_K of $\overline{f}$ and f. Then the dual graph of $\overline{Y}_K \setminus Y_K$ is the same linear chain of smooth rational curves as the closed fibers $\overline{Y}_t \setminus Y_t$. Write $\overline{Y}_0 \setminus Y_0 = \sum_{i=1}^{r} C_i$. If C_i and C_j meet for $i \neq j$, then the intersection point $C_i \cap C_j$ moves on the intersection curve $S_i \cdot S_j$. Since any minimal normal completion of $\mathbb{A}^2$ can be brought to $\mathbb{P}^2$ by blowing ups and downs with centers on the boundary divisor, we can blow up simultaneously the intersection curves and blow down the proper transforms of the S_i according to the blowing ups and downs on $\overline{Y}_0$. Here we note that the begining center of blowing up is a point on a (0)-curve C_1. In this case, we choose a suitable cross-section on the irreducible component S_1 which is a $\mathbb{P}^1$-bundle in the Zariski topology because $\dim T = 1$. Note that if T is irrational, then the chosen cross-section may meet the intersection curves on S_1 with other components of S. Then we shrink T so that the cross-section does not meet the intersection curves. If T is rational, S_1 is a trivial $\mathbb{P}^1$-bundle, hence we do not need the procedure of shrinking T. Thus we may assume that, for every $t \in T$, $\overline{Y}_t$ is isomorphic to $\mathbb{P}^2$ and $\overline{Y}_t \setminus Y_t$ is a single curve C_t with $(C_t)^2 = 1$. This implies that $\overline{Y}_K \cong \mathbb{P}^2_K$ and $Y_K \cong \mathbb{A}^2_K$. □

Remark 2.10. In the second proof of Theorem 2.8, when we have to shrink the base curve T, we replace T by a Zariski open neighborhood of the point t_0 of T. If $(Y, \overline{Y}, S, \overline{f}, t_0)$ is a log deformation of $(\overline{Y}_0, D_0, Y_0)$ such that $Y_t \cong \mathbb{A}^2$ for every closed point $t \in T$, we can argue in the same fashion by replacing t_0 by every closed point $t \in T$. This implies that T has an open covering $T = \cup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is a trivial $\mathbb{A}^2$-bundle over U_i. Hence Y is an $\mathbb{A}^2$-bundle over T. □

In connection with Theorem 2.8, we can pose the following

Conjecture 2.11. *Let K be a field of characteristic zero and let X be a smooth affine surface defined over K. Suppose that $X \otimes_K \overline{K}$ has an $\mathbb{A}^1$-fibration of affine type, where $\overline{K}$ is an algebraic closure of K. Then X has an $\mathbb{A}^1$-fibration of affine type.*

If we consider an $\mathbb{A}^1$-fibration of complete type, an example of Dubouloz-Kishimoto gives a counter-example to the above conjecture (see Theorem 5.1).

3. Topological arguments instead of Hilbert schemes

In this section we will briefly indicate topological proofs of some of the results in the section two. The use of topological arguments would make the cumbersome geometric arguments more transparent for the people who do not appreciate the heavy machinery like Hilbert scheme.

We will use the following basic theorem due to Ehresmann [29, Chapter V, Prop. 6.4].

Theorem 3.1. *Let M be a connected differentiable manifold, S a closed submanifold, $f : M \to N$ a proper differentiable map such that the tangent maps corresponding to f and $f|_S : S \to N$ are surjective at any point in M and S. Then $f|_{M \setminus S} : M \setminus S \to N$ is a locally trivial fiber bundle with respect to the base N.*

Note that the normal bundle of any fiber of f is trivial. We can give a proof of Ehresmann's theorem using this observation, and the well-known result from differential topology that given a compact submanifold S of a C^∞ manifold X there are arbitrarily small tubular neighborhoods of S in X which are diffeomorphic to neighborhoods of S in the total space of normal bundle of S in X [1, Chapter II, Theorem 11.14].

Now let $\overline{f} : \overline{Y} \to T$ be a smooth projective morphism from a smooth algebraic threefold onto a smooth algebraic curve T. Let $\overline{Y}_t = \overline{f}^{-1}(t)$ be the fiber over $t \in T$. Let S be a simple normal crossing divisor on

$\overline{Y}$ such that $D_t := S \cap \overline{Y}_t$ is a simple normal crossing divisor for each $t \in T$ and $Y_t := \overline{Y}_t \setminus D_t$ is affine for each t.

We can assume that $\overline{f} : \overline{Y} \to T$ has the property that the tangent map is surjective at each point. It follows from Ehresmann's theorem that all the surfaces $\overline{Y}_t$ are mutually diffeomorphic. In particular, they have the same topological invariants like the fundamental group π_1 and the Betti number b_i. By shrinking T if necessary, we will assume that the restricted map $\overline{f} : S_i \to T$ is smooth for each i. For fixed i and t_0 the intersection $S_i \cap \overline{Y}_{t_0}$ is a disjoint union of smooth, compact, irreducible curves. Let $C_{t_0,i}$ be one of these irreducible curves. Then for each t which is close to t_0, there is an irreducible curve $C_{t,i}$ in $S_i \cap \overline{Y}_t$ and suitable tubular neighborhoods of $C_{t_0,i}$, $C_{t,i}$ in $\overline{Y}_{t_0}$, $\overline{Y}_t$ respectively are diffeomorphic by Ehresmann's theorem. This implies that $C_{t_0,i}^2$ in $\overline{Y}_{t_0}$ and $C_{t,i}^2$ in $\overline{Y}_t$ are equal. This proves that the weighted dual graphs of the curves D_t in $\overline{Y}_t$ are the same for each t.

Recall that if X is a smooth projective surface with a smooth rational curve $C \subset X$ such that $C^2 = 0$ then C is a fiber of a $\mathbb{P}^1$-fibration on X. This follows easily from Riemann-Roch theorem if X is rational. If the irregularity $q(X) > 0$ then the Albanese morphism $X \to \text{Alb}(X)$ gives a $\mathbb{P}^1$-fibration on X with C as a fiber. By the above discussion the fiber surfaces $\overline{Y}_t$ have the same irregularity.

Suppose that $\overline{Y}_0$ has an $\mathbb{A}^1$-fibration of affine type $f : Y_0 \to B$. If $\overline{f} : \overline{Y}_0 \to \overline{B}$ is an extension of f to a smooth completion of Y_0 then, after simultaneous blowing ups and downs along the fibers of $\overline{f}$, we may asssume that $D_0 := \overline{Y}_0 \setminus Y_0$ contains at least one (0)-curve which is a tip, i.e., the end component of a maximal twig of D_0. Since D_t and D_0 have the same weighted dual graphs D_t also contains a (0)-curve which is a tip of D_t. Hence, Y_t also has an $\mathbb{A}^1$-fibration of affine type. This proves the assertion (2) in Lemma 2.2.

We can also shorten the part of showing the invariance of the boundary weighted graphs in the second proof of Theorem 2.8. Suppose now that $f : Y \to T$ is a fibration on a smooth affine threefold Y onto a smooth curve T such that every scheme-theoretic fiber of f is isomorphic to $\mathbb{A}^2$. We can embed Y in a smooth projective threefold $\overline{Y}$ such that f extends to a morphism $\overline{f} : \overline{Y} \to T$. By shrinking T we can assume that $\overline{f}$ is smooth, each irreducible component S_i of $\overline{Y} \setminus Y$ intersects each $\overline{Y}_t$ transversally, etc. By the above discussions, each $D_t := \overline{Y}_t \setminus Y_t$ has the same weighted dual graph. Since Y_t is isomorphic to $\mathbb{A}^2$, we can argue as in the second proof of Theorem 2.8 using the result of Ramanujam-Morrow to conclude that f is a trivial $\mathbb{A}^2$-bundle on a non-empty Zariski-open subset of T. This observation applies also to the proof of Theorem 4.6.

4. Deformations of ML_0 surfaces

For $i = 0, 1, 2$, an ML_i surface is by definition a smooth affine surface X such that the Makar-Limanov invariant $\mathrm{ML}(X)$ has transcendence degree i over k [10]. In this section, we assume that the ground field k is the complex field $\mathbb{C}$. Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a family satisfying the conditions of Lemma 2.2. Let $D_0 = S \cap \overline{Y}_0$.

Lemma 4.1. *Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(\overline{Y}_0, D_0, Y_0)$. Assume that D_0 is a tree of smooth rational curves satisfying one of the following conditions.*

(i) *D_0 contains an irreducible component C_1 such that $(C_1^2) \geq 0$.*
(ii) *D_0 contains a (-1) curve which meets more than two other components of D_0.*

Then the following assertions hold after changing T by an étale finite covering of an open set of T if necessary.

(1) *Every irreducible component of D_0 deforms along the fibers of $\overline{f}$. Namely, if $D_0 = \sum_{i=1}^r C_i$ is the irreducible decomposition, then, for every $1 \leq i \leq r$, there exists an irreducible component S_i of S such that $\overline{f}|_{S_i} : S_i \to T$ has the fiber $(\overline{f}|_{S_i})^{-1}(t_0) = C_i$. Furthermore, $S = \sum_{i=1}^r S_i$.*
(2) *For $t \in T$, let $C_{i,t} = (\overline{f}|_{S_i})^{-1}(t)$. Then $D_t = \sum_{i=1}^r C_{i,t}$ and D_t has the same weighted graph on $\overline{Y}_t$ as D_0 does on $\overline{Y}_0$.*
(3) *For every i, $\overline{f}|_{S_i} : S_i \to T$ is a trivial $\mathbb{P}^1$-bundle over T.*

Proof. The argument is analytic locally almost the same as in the proof for the assertion (2) of Lemma 2.2. Consider the deformation of C_1 along the fibers of $\overline{f}$, which moves along the fibers because $(C_1^2) \geq -1$. Then the components of D_0 which are adjacent to C_1 also move along the fibers of $\overline{f}$. Once these components of D_0 move, then the components adjacent to these components move along the fibers of $\overline{f}$. Since D_0 is connected because Y_0 is affine, all the components of D_0 move along the fibers of $\overline{f}$. If S contains an irreducible component which does not intersect $\overline{Y}_0$, it is a fiber component of $\overline{f}$. Then we remove the fiber by shrinking T. This proves the assertion (1).

Let $S = \sum_{i=1}^r S_i$ be the irreducible decomposition of S. As shown in (1), $S_i \cap \overline{Y}_0 \neq \emptyset$ for every i. Then $S_i \cap \overline{Y}_t \neq \emptyset$ as well by the argument in the proof of Lemma 2.2.

Note that $((S_i \cdot \overline{Y}_t)^2)_{\overline{Y}_t} = (S_i^2 \cdot \overline{Y}_t) = (S_i^2 \cdot \overline{Y}_0) = ((S_i \cdot \overline{Y}_0)^2)_{\overline{Y}_0}$ because $\overline{Y}_t$ is algebraically equivalent to $\overline{Y}_0$. Hence D_0 and D_t have the same dual graphs. □

In order to prove the following result, we use Ehresmann's theorem, which is Theorem 3.1.

Lemma 4.2. *Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(\overline{Y}_0, D_0, Y_0)$ which satisfy the same conditions as in Lemma 4.1. Assume further that $p_g(\overline{Y}_0) = q(\overline{Y}_0) = 0$. Then the following assertions hold.*

1. $\mathrm{Pic}\,(Y_t) \cong \mathrm{Pic}\,(Y_0)$ *for every $t \in T$.*
2. $\Gamma(Y_t, \mathcal{O}^*_{Y_t}) \cong \Gamma(Y_0, \mathcal{O}^*_{Y_0})$ *for every $t \in T$.*

Proof. Since p_g and q are deformation invariants, we have $p_g(\overline{Y}_t) = q(\overline{Y}_t) = 0$ for every $t \in T$. The exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\overline{Y}_t} \xrightarrow{\exp} \mathcal{O}^*_{\overline{Y}_t} \longrightarrow 0$$

induces an exact sequence

$$H^1(\overline{Y}_t, \mathcal{O}_{\overline{Y}_t}) \to H^1(\overline{Y}_t, \mathcal{O}^*_{\overline{Y}_t}) \to H^2(\overline{Y}_t; \mathbb{Z}) \to H^2(\overline{Y}_t, \mathcal{O}_{\overline{Y}_t})$$

Since $p_g(\overline{Y}_t) = q(\overline{Y}_t) = 0$, we have an isomorphism

$$H^1(\overline{Y}_t, \mathcal{O}^*_{\overline{Y}_t}) \cong H^2(\overline{Y}_t; \mathbb{Z}) \ .$$

Now consider the canonical homomorphism $\theta_t : H_2(D_t; \mathbb{Z}) \to H_2(\overline{Y}_t; \mathbb{Z})$, where $H_2(\overline{Y}_t; \mathbb{Z}) \cong H^2(\overline{Y}_t; \mathbb{Z}) = \mathrm{Pic}(\overline{Y}_t)$ by the Poincaré duality. Then $\mathrm{Coim}\,\theta_t = \mathrm{Pic}\,(Y_t)$ and $\mathrm{Ker}\,\theta_t = \Gamma(Y_t, \mathcal{O}^*_{Y_t})/k^*$.

Let N be a nice tubular neighborhood of S with boundary in $\overline{Y}$. The smooth morphism $\overline{f} : \overline{Y} \to T$ together with its restriction on the $(N, \partial N)$ gives a proper differential mapping which is surjective and submersive. By Theorem 3.1, it is differentiably a locally trivial fibration. Namely, there exists a small disc U of t_0 in T and a diffeomorphism $\varphi_0 : \overline{Y}_0 \times U \xrightarrow{\approx} (\overline{f})^{-1}(U)$ such that its restriction induces a diffeomorphism

$$\varphi_0 : (N \cap \overline{Y}_0) \times U \xrightarrow{\approx} (\overline{f}|_N)^{-1}(U) \ .$$

For $t \in U$, noting that U is contractible and hence $H_2(\overline{Y}_0 \times U; \mathbb{Z}) = H_2(\overline{Y}_0; \mathbb{Z})$ and $H_2((N \cap \overline{Y}_0) \times U; \mathbb{Z}) = H_2(N \cap \overline{Y}_0; \mathbb{Z})$, the inclusions $\overline{Y}_t \hookrightarrow (\overline{f})^{-1}(U)$ and $N \cap \overline{Y}_0 \hookrightarrow (\overline{f}|_N)^{-1}(U)$ induces compatible isomorphisms

$$p_t : H_2(\overline{Y}_t; \mathbb{Z}) \to H_2((\overline{f})^{-1}(U); \mathbb{Z}) \xrightarrow{(\varphi^{-1})_*} H_2(\overline{Y}_0 \times U; \mathbb{Z}) = H_2(\overline{Y}_0; \mathbb{Z})$$

and its restriction $q_t : H_2(N \cap \overline{Y}_t; \mathbb{Z}) \xrightarrow{\sim} H_2(N \cap \overline{Y}_0; \mathbb{Z})$. Since S and hence D_t are strong deformation retracts of N and $N \cap \overline{Y}_t$ respectively, the isomorphism q_t induces an isomorphism $r_t : H_2(D_t; \mathbb{Z}) \xrightarrow{\sim} H_2(D_0; \mathbb{Z})$ such that the following diagram

$$\begin{array}{ccc} H_2(D_t;\mathbb{Z}) & \xrightarrow{\theta_t} & H_2(\overline{Y}_t;\mathbb{Z}) \\ {\scriptstyle r_t}\downarrow & & \downarrow{\scriptstyle p_t} \\ H_2(D_0;\mathbb{Z}) & \xrightarrow[\theta_0]{} & H_2(\overline{Y}_0;\mathbb{Z}) \end{array}$$

This implies that Pic $(Y_t) \cong$ Pic (Y_0) and $\Gamma(Y_t, \mathcal{O}^*_{Y_t}) \cong \Gamma(Y_0, \mathcal{O}^*_{Y_0})$. If t is an arbitrary point of T, we choose a finite sequence of points $\{t_0, t_1, \ldots, t_n = t\}$ such that t_i is in a small disc U_{i-1} around t_{i-1} $(1 \le i \le n)$ for which we can apply the above argument. □

Remark 4.3. By a result of W. Neumann [25, Theorem 5.1], if X is a normal affine surface, D an SNC divisor at infinity of X which does not contain any (-1)-curve meeting at least three other components of D and all whose maximal twigs are smooth rational curves with self-intersections ≤ -2, then the boundary 3-manifold of a nice tubular neighborhood N of D determines the dual graph of D. If we use the local differentiable triviality of a tubular neighborhood N, this result of Neumann shows that the weighted dual graph of D_t is deformation invariant.

According to [10, Lemmas 1.2, 1.4], we have the following property and characterization of ML_0-surface.

Lemma 4.4. *Let X be a smooth affine surface and let V be a minimal normal completion of X. Then the following assertions hold.*

(1) *X is an ML_0-surface if and only if $\Gamma(X, \mathcal{O}^*_X) = k^*$ and the dual graph of the boundary divisior $D := V - X$ is a linear chain of smooth rational curves.*
(2) *If X is an ML_0-surface, X has an $\mathbb{A}^1$-fibration, and any $\mathbb{A}^1$-fibration $\rho : X \to B$ has base curve either $B \cong \mathbb{P}^1$ or $B \cong \mathbb{A}^1$. If $B \cong \mathbb{P}^1$, ρ has at most two multiple fibers, and if $B \cong \mathbb{A}^1$, it has at most one multiple fiber.*

The following result is a direct consequence of the above lemmas.

Theorem 4.5. *Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(\overline{Y}_0, D_0, Y_0)$, where Y_0 is an ML_0-surface. Then Y_t is an ML_0-surface for every $t \in T$.*

Proof. If $S \cap \overline{Y}_t$ contains a (-1) curve, then it deforms along the fibers of $\overline{f}$ after an étale finite base change of T, and these (-1) curves are contracted simultaneously by Lemma 2.1. Hence we may assume that $\overline{Y}_t$ is a minimal normal completion of Y_t for every $t \in T$. By Lemma 4.4, $D_0 := S \cap \overline{Y}_0$ is a linear chain of smooth rational curves. Hence $D_t := S \cap \overline{Y}_t$ is also a linear chain of smooth rational curves. By Lemma 4.2, $\Gamma(Y_t, \mathcal{O}^*_{Y_t}) = k^*$ for every $t \in T$ because $\Gamma(Y_0, \mathcal{O}^*_{Y_0}) = k^*$. So, Y_t is an ML_0-surface by Lemma 4.4. □

A smooth affine surface X is, by definition, an *affine pseudo-plane* if it has an $\mathbb{A}^1$-fibration of affine type $p : X \to \mathbb{A}^1$ admitting at most one multiple fiber of the form $m\mathbb{A}^1$ as a singular fiber (see [23] for the definition and relevant results). An affine pseudo-plane is a $\mathbb{Q}$-homology plane, its Picard group is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ and there are no non-constant invertible elements. An ML_0-surface is an affine pseudo-plane if the Picard number is zero.

If $\overline{X}$ is a minimal normal completion of an affine pseudo-plane X, the boundary divisor $D = \overline{X} - X$ is a tree of smooth rational curves, which is not necessarily a linear chain. By blowing-ups and blowing-downs with centers on the boundary divisor D, we can make the completion $\overline{X}$ satisfy the following conditions [23, Lemma 1.7].

(i) There is a $\mathbb{P}^1$-fibration $\overline{p} : \overline{X} \to \mathbb{P}^1$ which extends the $\mathbb{A}^1$-fibration $p : X \to \mathbb{A}^1$.

(ii) The weighted dual graph of D is

$$\underset{\ell}{(0)} \;—\; \underset{M}{(0)} \;—\; A$$

(iii) There is a (-1) curve F_0 such that $F_0 \cap X \cong \mathbb{A}^1$ and the union F_0 — A is contractible to a smooth rational curve meeting the image of the component M.

Note that X is an ML_0-surface if and only if A is a linear chain.

If we are given a log deformation $(Y, \overline{Y}, S, \overline{f}, t_0)$ of the triple $(\overline{Y}_0, D_0, Y_0)$, it follows by Ehresmann's fibration theorem that p_g and the irregularity q of the fiber $\overline{Y}_t$ is independent of t. Furthermore, by Lemma 2.2, Y_t has an $\mathbb{A}^1$-fibration if Y_0 has an $\mathbb{A}^1$-fibration. So, we can expect that Y_t is an affine pseudo-plane if so is Y_0. Indeed, we have the following result.

Theorem 4.6. *Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(\overline{Y}_0, D_0, Y_0)$. Assume that Y_0 is an affine pseudo-plane. Then $f : Y \to T$ is a trivial bundle with fiber Y_0 after shrinking T if necessary.*

Proof. Consider the completion $\overline{Y}_0$ of Y_0. We may assume that $\overline{Y}_0$ is a minimal normal completion of Y_0. Note that every fiber Y_t has an $\mathbb{A}^1$-fibration of affine type by Lemma 2.2. As in the proof of Lemma 2.2, (2), by performing simultaneous (i.e., along the fibers of $\overline{f}$) blowing-ups and blowing-downs on the boundary S, we may assume that Y_0 has an $\mathbb{A}^1$-fibration which extends to a $\mathbb{P}^1$-fibration on $\overline{Y}_0$ and that the boundary divisor D_0 has the weighted dual graph ℓ — M — A as specified in the condition (ii) above. To perform a simultaneous blowing-up, we may have to choose as the center a cross-section on an irreducible component S_i which is a $\mathbb{P}^1$-bundle over T. If such a cross-section happens to intersect the curve $S_i \cap S_j$ with another component S_j, we shrink T to avoid this intersection (see the remark in the second proof of Theorem 2.8). Note that the interior Y (more precisely, the inverse image of f of the shrunken T) is not affected under these operations. Then the (0) curve ℓ defines a $\mathbb{P}^1$-fibration $\varphi : \overline{Y} \to V$ (see Lemma 2.1, (3)). In particular, ℓ moves in an irreducible component, say S_{-1}, of S. The (0) curve M moves along the fibers of $\overline{f}$ in an irreducible component, say S_0, of S. By Lemma 4.1, the curves in A move along the fibers of $\overline{f}$ and fill out the irreducible components $S_1, \ldots, S_r$ of S. Hence $S = S_{-1} \cup S_0 \cup S_1 \cup \cdots \cup S_r$ and $D_t = S \cdot \overline{Y}_t$ has the same weighted dual graph as D_0.

Now consider a (-1) curve F_0 on $\overline{Y}_0$. By Lemma 2.1, F_0 moves along the fibers of $\overline{f}$ and fills out a smooth irreducible divisor F which meets transversally an irreducible component S_i $(1 \le i \le r)$. In fact, $(S_i \cdot F \cdot \overline{Y}_t) = (S_i \cdot F \cdot \overline{Y}_0) = 1$. Let S_1 be the component of S meeting S_0. Let $F_t = F \cap \overline{Y}_t$ and $S_{j,t} = S_j \cap \overline{Y}_t$ for every $t \in T$. Then $F_t + \sum_{j=2}^{r} S_{j,t}$ is contractible to a smooth point P_t lying on $S_{1,t}$. After performing simultaneous elementary transformations on the fiber ℓ which is the fiber at infinity of the $\mathbb{A}^1$-fibration of the affine pseudo-plane Y_t, we may assume that P_t is the intersection point $S_{0,t} \cap S_{1,t}$. By applying Lemma 2.1, (4) repeatedly, we can contract F and the components $S_2, \ldots, S_r$ simultaneously. Let $\overline{Z}$ be the threefold obtained from $\overline{Y}$ by these contractions. Then $\overline{Z}$ has a $\mathbb{P}^1$-fibration $\psi : \overline{Z} \to V$ and the image of S_0 is a cross-section. Let $g = \sigma \cdot \psi : \overline{Z} \xrightarrow{\psi} V \xrightarrow{\sigma} T$ (see Lemma 2.1, (3) for the notations). For every $t \in T$, $\overline{Z}_t := g^{-1}(t)$ is a minimal $\mathbb{P}^1$-bundle with a cross-section $S_{1,t}$. Since $(S_{1,t})^2 = 0$, $\overline{Z}_t$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\overline{Z}$ is a trivial $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over T after shrinking T if necessary. In fact, $\overline{Z}$ with the images of S_0 and S_{-1} removed is a deformation of $\mathbb{A}^2$, which is locally trivial in the Zariski topology by Theorem 2.8. We may assume that $\psi : \overline{Z} \to V$ is the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times T$ onto the second and the third factors. Choose a section

$\overline{S}'_0$ of ψ which is disjoint from the image $\overline{S}_0$ of S_0. Then there is a non-trivial G_m-action on $\overline{Z}$ along the fibers of ψ which has $\overline{S}_0$ and $\overline{S}'_0$ as the fixed point locus.

Now reverse the contractions $\overline{Y} \to \overline{Z}$. The center of the first simultaneous blowing-up with center $S_0 \cap S_1$ and the centers of the consecutive simultaneous blowing-ups except for the blowing-up which produces the component F are G_m-fixed because the blowing-ups are fiberwise sub-divisional. Only the center Q_t of the last blowing-up on $\overline{Y}_t$ is non-subdivisional. Let

$$\varphi : \overline{Y} \xrightarrow{\sigma} \overline{Y}_1 \xrightarrow{\sigma_1} \overline{Z}$$

be the factorization of φ where σ is the last non-subdivisional blowing-up. By the construction, the natural T-morphism $\overline{f}_1 : \overline{Y}_1 \to T$ is a trivial fibration with fiber $(\overline{Y}_1)_0 = \overline{f}_1^{-1}(t_0)$. Then there exists an element $\{\rho_t\}_{t\in T}$ of $G_m(T)$ such that $\rho_t(Q_{t_0}) = Q_t$ for every $t \in T$ after shrinking T if necessary. Here note that the G_m-action is nontrivial on the component with the point Q_t thereon, for otherwise the G_m-action is trivial from the beginning. Then these $\{\rho_t\}_{t\in T}$ extends to a T-isomorphism $\widetilde{\rho} : \overline{Y}_0 \times T \to \overline{Y}$, which induces a T-isomorphism $Y_0 \times T \to Y$. Hence Y is trivial. □

The following result is a generalization of Theorem 2.8.

Theorem 4.7. *Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold Y to a smooth affine curve T. Assume that the fiber Y_t is an affine pseudo-plane for every closed point of T. Let t_0 be a general closed point of T. Then the generic fiber Y_K of f is isomorphic to the affine pseudo-plane $Y_0 = f^{-1}(t_0) \otimes_k K$, where $K = k(T)$. Hence $f : Y \to T$ is a trivial bundle over T with fiber Y_0 after replacing T by an open neighborhood of t_0 if necessary.*

Proof. Let $f : Y \to T$ be as in Theorem 4.7. As in the second proof of Theorem 2.8, we can find a relative completion $\overline{Y}$ of Y such that $\overline{Y}$ is smooth and f extends to a smooth projective morphism. Furthermore, choosing a suitable point $t_0 \in T$ and considering the triple $(\overline{f}^{-1}(t_0), S \cap \overline{f}^{-1}(t_0), f^{-1}(t_0))$ as $(\overline{Y}_0, D_0, Y_0)$, we can find a log deformation $(Y, \overline{Y}, S, \overline{f}, t_0)$ of the triple $(\overline{Y}_0, D_0, Y_0)$. To obtain this setting, we may have to shrink T to a smaller open set of T. Since we assume that Y_t is an affine pseudo-plane for every $t \in T$, this situation is realizable. Then the result follows from Theorem 4.6. □

5. Deformations of $\mathbb{A}^1$-fibrations of complete type

In the setting of Theorem 2.6, if the $\mathbb{A}^1$-fibration of a general fiber Y_t is of complete type, we do not have the same conclusion. This case is treated in a recent work of Dubouloz and Kishimoto [3]. We consider this case by taking the same example of cubic surfaces in $\mathbb{P}^3$ and explain how it is affine-uniruled.

Taking a cubic hypersurface as an example, we first observe the behavior of the log Kodaira dimension for a flat family of smooth affine surfaces. Let $\overset{\vee}{\mathbb{P}^3}$ be the dual projective 3-space whose points correspond to the hyperplanes of $\mathbb{P}^3$. We denote it by T. Let S be a smooth cubic hypersurface in $\mathbb{P}^3$ and let $\mathcal{W} = S \times T$ which is a codimension one subvariety of $\mathbb{P}^3 \times T$. Let $\mathcal{H}$ be the universal hyperplane in $\mathbb{P}^3 \times T$, which is defined by $\xi_0 X_0 + \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3 = 0$, where (X_0, X_1, X_2, X_3) and $(\xi_0, \xi_1, \xi_2, \xi_3)$ are respectively the homogeneous coordinates of $\mathbb{P}^3$ and T. Let $\mathcal{D}$ be the intersection of $\mathcal{W}$ and $\mathcal{H}$ in $\mathbb{P}^3 \times T$. Let $\pi : \mathcal{W} \to T$ be the projection and let $\pi_{\mathcal{D}} : \mathcal{D} \to T$ be the restriction of π onto $\mathcal{D}$. Then π and $\pi_{\mathcal{D}}$ are the flat morphism. For a closed point $t \in T$, $\mathcal{W}_t = \pi^{-1}(t)$ is identified with S and $\mathcal{D}_t = \pi_{\mathcal{D}}^{-1}(t)$ is the hyperplane section $S \cap \mathcal{H}_t$ in $\mathbb{P}^3$, where $\mathcal{H}_t$ is the hyperplane $\tau_0 X_0 + \tau_1 X_1 + \tau_2 X_2 + \tau_3 X_3 = 0$ with $t = (\tau_0, \tau_1, \tau_2, \tau_3)$. Let $\mathcal{X} = \mathcal{W} \setminus \mathcal{D}$ and $p : \mathcal{X} \to T$ be the restriction of π onto $\mathcal{X}$. Then $\mathcal{X}_t = p^{-1}(t)$ is an affine surface $S \setminus (S \cap \mathcal{H}_t)$.

Since S is smooth, the following types of $S \cap \mathcal{H}_t$ are possible. In the following, $F = 0$ denotes the defining equation of S and $H = 0$ does the equation for $\mathcal{H}_t$.

(1) A smooth irreducible plane curve of degree 3.
(2) An irreducible nodal curve, e.g., $F = X_0(X_1^2 - X_2^2) - X_2^3 + X_0^2 X_3 + X_3^3$ and $H = X_3$.
(3) An irreducible cuspidal curve, e.g., $F = X_0 X_1^2 - X_2^3 + X_3(X_0^2 + X_1^2 + X_2^2 + X_3^2)$ amd $H = X_3$.
(4) An irreducible conic and a line which meets in two points transversally or in one point with multiplicity two. In fact, let ℓ and D be respectively a line and an irreducible conic in $\mathbb{P}^2$ meeting in two points Q_1, Q_2, where Q_1 is possibly equal to Q_2. Let C be a smooth cubic meeting ℓ in three points P_i $(1 \leq i \leq 3)$ and D in six points P_i $(4 \leq i \leq 9)$, where the points P_i are all distinct and different from Q_1, Q_2. Choose two points P_1, P_2 on ℓ and four points P_i $(4 \leq i \leq 7)$ on D. Let $\sigma : S \to \mathbb{P}^2$ be the blowing-up of these six points. Let ℓ', D' and C' be the proper transforms of ℓ, D and C'. Then S is a cubic hypersurface in $\mathbb{P}^3$

and $K_S \sim -C'$. Since $\ell' + D' \sim C'$, it is a hyperplane section of S with respect to the embedding $\Phi_{|C'|} : S \hookrightarrow \mathbb{P}^3$.

(5) Three lines which are either meeting in one point or not. Let ℓ_i $(1 \le i \le 3)$ be the lines. Let $Q_1 = \ell_1 \cap \ell_3$ and $Q_2 = \ell_2 \cap \ell_3$. In the setting of (4) above, we consider $\ell = \ell_3$ and $D = \ell_1 + \ell_2$. So, if $Q_1 = Q_2$, three lines meet in one point. Choose a smooth cubic C meeting three lines in nine distinct points P_i $(1 \le i \le 9)$ other than Q_1, Q_2. Choose six points from the P_i, two points lying on each line. Then consider the blowing-up in these six points. The rest of the construction is the same as above.

Note that if S is smooth $S \cap \mathcal{H}_t$ cannot have a non-reduced component. In fact, the non-reduced component is a line in $\mathcal{H}_t$. Hence we may write the defining equation of S as

$$F = X_0^2(aX_1 + X_0) + X_3 G(X_0, X_1, X_2, X_3) = 0,$$

where $G = G(X_0, X_1, X_2, X_3)$ is a quadratic homogeneous polynomial and $a \in k$. We understand that $a = 0$ if the non-reduced component has multiplicity three. By the Jacobian criterion, it follows that S has singularities at the points $G = X_0 = X_3 = 0$.

The affine surface $\mathcal{X}_t$ has log Kodaira dimension 0 in the cases (1), (2), (4) with the conic and the line meeting in two distinct points and (5) with non-confluent three lines, and $-\infty$ in the rest of the cases. Although $p : \mathcal{H} \to T$ is a flat family of affine surfaces, the log Kodaira dimension drops to $-\infty$ exactly at the points $t \in T$ where the boundary divisor $S \cap \mathcal{H}_t$ is not a divisor with normal crossings. This accords with a result of Kawamata concerning the invariance of log Kodaira dimension under deformations (cf. [17]).

If $\overline{\kappa}(\mathcal{X}_t) = -\infty$, then $\mathcal{X}_t$ has an $\mathbb{A}^1$-fibration. We note that if $\overline{\kappa}(\mathcal{X}_t) = 0$ then $\mathcal{X}_t$ has an $\mathbb{A}^1_*$-fibration. In fact, we consider the case where the boundary divisor $\mathcal{D}_t$ is a smooth cubic curve. Then S is obtained from $\mathbb{P}^2$ by blowing up six points P_i $(1 \le i \le 6)$ on a smooth cubic curve C. Choose four points P_1, P_2, P_3, P_4 and let Λ be a linear pencil of conics passing through these four points. Let $\sigma : S \to \mathbb{P}^2$ be the blowing-up of six points P_i $(1 \le i \le 6)$. The proper transform $\sigma'\Lambda$ defines a $\mathbb{P}^1$-fibration $f : S \to \mathbb{P}^1$ for which the proper transform $C' = \sigma'(C)$ is a 2-section. Since $\mathcal{X}_t$ is isomorphic to $S \setminus C'$, $\mathcal{X}_t$ has an $\mathbb{A}^1_*$-fibration.

Looking for an $\mathbb{A}^1$-fibration in the case $\overline{\kappa}(\mathcal{X}_t) = -\infty$ is not an easy task. Consider, for example, the case where $X = \mathcal{X}_t$ is obtained as $S \setminus (Q \cup \ell)$, where Q is a smooth conic and ℓ is a line in $\mathbb{P}^2$ which meet in one point with multiplicity two. As explained in the above, such an X is obtained from $\mathbb{P}^2$ by blowing up six points $P_1, \ldots, P_6$ such that P_1, P_2 lie on a line $\widetilde{\ell}$ and P_3, P_4, P_5, P_6 are points on a conic $\widetilde{Q}$. Then the proper transforms on S of $\widetilde{\ell}, \widetilde{Q}$ are ℓ, Q. Consider the linear pencil $\widetilde{\Lambda}$ on $\mathbb{P}^2$ spanned by $2\widetilde{\ell}$ and $\widetilde{Q}$. Then a general member of Λ is a smooth conic meeting $\widetilde{Q}$ in one point $\widetilde{Q} \cap \widetilde{\ell}$ with multiplicity four. The proper transform Λ of $\widetilde{\Lambda}$ on S defines an $\mathbb{A}^1$-fibration on X.

The following result of Dubouloz-Kishimoto except for the assertion (4) was orally communicated to one of the authors (see [3]).

Theorem 5.1. *Let S be a cubic hypersurface in $\mathbb{P}^3$ with a hyperplane section $S \cap H$ which consists of a line and a conic meeting in one point with multiplicity two. Let $Y = \mathbb{P}^3 \setminus S$ which is a smooth affine threefold. Then the following assertions hold.*

1. $\overline{\kappa}(Y) = -\infty$.
2. *Let $f : Y \to \mathbb{A}^1$ be a fibration induced by the linear pencil on $\mathbb{P}^3$ spanned by S and $3H$. Then a general fiber Y_t of f is a cubic hypersurface S_t minus $Q \cup \ell$, where Q is a conic and ℓ is a line which meet in one point with multiplicity two. Hence $\overline{\kappa}(Y_t) = -\infty$ and Y_t has an $\mathbb{A}^1$-fibration.*
3. *Y has no $\mathbb{A}^1$-fibration.*
4. *There is a finite covering T' of $\mathbb{A}^1$ such that the normalization of $Y \times_{\mathbb{A}^1} T'$ has an $\mathbb{A}^1$-fibration.*

Proof. (1) Since $K_{\mathbb{P}^3} + S \sim -4H + 3H = -H$, it follows that $\overline{\kappa}(Y) = -\infty$.

(2) The pencil spanned by S and $3H$ has base locus $Q \cup \ell$ and its general member, say S_t, is a cubic hypersurface containing $Q \cup \ell$ as a hyperplane section. It is clear that $S_t \setminus (Q \cup \ell) = Y_t$. Hence, as explained above, Y_t has an $\mathbb{A}^1$-fibration.

(3) Let $\tau : \widetilde{S} \to \mathbb{P}^3$ be the cyclic triple covering of $\mathbb{P}^3$ ramified totally over the cubic hypersurface S. Then $\widetilde{S}$ is a cubic hypersurface in $\mathbb{P}^4$ and $\tau^*(S) = 3\widetilde{H}$, where $\widetilde{H}$ is a hyperplane in $\mathbb{P}^4$. The restriction of τ onto $Z := \widetilde{S} \setminus \widetilde{S} \cap \widetilde{H}$ induces a finite étale covering $\tau_Z : Z \to Y$. Suppose that Y has an $\mathbb{A}^1$-fibration $\varphi : Y \to T$. Then T is a rational surface. Since τ_Z is finite étale, this $\mathbb{A}^1$-fibration φ lifts up to an $\mathbb{A}^1$-fibration $\widetilde{\varphi} : Z \to \widetilde{T}$. By [2], $\widetilde{S}$ is unirational and irrational. Hence $\widetilde{T}$ is a rational surface. This implies that Z is a rational threefold. This is a contradiction because $\widetilde{S}$ is irrational.

(4) There is an open set T of $\mathbb{A}^1$ such that the restriction of f onto $f^{-1}(T)$ is a smooth morphism onto T. By abuse of the notations, we denote $f^{-1}(T)$ by Y anew and the restriction of f onto $f^{-1}(T)$ by f. Hence $f : Y \to T$ is a smooth morphism. Let $K = k(t)$ be the function field of T and let Y_K be the generic fiber. Let $\overline{K}$ be an algebraic

closure of K. Then $Y_{\overline{K}} := Y_K \otimes_K \overline{K}$ is identified with $S_{\overline{K}} \setminus (Q \cup \ell)$, where $S_{\overline{K}}$ is a cubic hypersurface in $\mathbb{P}^3_{\overline{K}}$ defined by $F_K = F_0 + tX_3^3 = 0$. Here t is a coordinate of $\mathbb{A}^1$ and (X_0, X_1, X_2, X_3) is a system of homogeneous coodinates of $\mathbb{P}^3$ such that $F_0(X_0, X_1, X_2, X_3) = 0$ is the defining equation of the cubic hypersurface S and the hyperplane H is defined by $X_3 = 0$. Then $Y_{\overline{K}}$ is obtained from $\mathbb{P}^2_{\overline{K}}$ by blowing up six $\overline{K}$-rational points in general position (two points on the image of ℓ and four points on the image of Q). As explained earlier, there is an $\mathbb{A}^1$-fibration on $Y_{\overline{K}}$ which is obtained from conics on $\mathbb{P}^2_{\overline{K}}$ belonging to the pencil spanned by Q and 2ℓ. This construction involves six points on $\mathbb{P}^2_{\overline{K}}$ to be blown up to obtain the cubic hypersurface $S_{\overline{K}}$ and four points (the point $Q \cap \ell$ and its three infinitely near points). Hence there exists a finite algebraic extension K'/K such that all these ponts are rational over K'. Let T' be the normalization of T in K'. Let $Y' = Y \otimes_K K'$. Then Y' has an $\mathbb{A}^1$-fibration. □

Based on the assertion (4) above, we propose the following conjecture.

Conjecture 5.2. *Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold Y onto a smooth affine curve T such that every closed fiber Y_t has an $\mathbb{A}^1$-fibration of complete type. Then there exists a finite covering T' of T such that the normalization of $Y \times_T T'$ has an $\mathbb{A}^1$-fibration.*

References

[1] G. Bredon, Topology and Geometry, Graduate Texts in Mathematics no. 139, Springer.

[2] C.H. Clemens and P.A. Griffiths, The intermediate Jacobian of the cubic threefold. Ann. of Math. (2) **95** (1972), 281–356.

[3] A. Dubouloz and T. Kishimoto, Log-uniruled affine varieties without cylinder-like open subsets, arXiv: 1212.0521, 2012.

[4] T. Fujita, On the topology of noncomplete algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), no. 3, 503–566.

[5] H. Flenner and M. Zaidenberg, $\mathbb{Q}$ -acyclic surfaces and their deformations, Classification of algebraic varieties (L'Aquila, 1992), 143–208, Contemp. Math. **162**, Amer. Math. Soc., 1994.

[6] H. Flenner, Sh. Kaliman and M. Zaidenberg, Deformation equivalence of affine ruled surfaces, arXiv:1305.5366v1, 23 May 2013.

[7] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert, Séminaire Bourbaki, 13e année, 1960/61, no. 221, 1–28.

[8] R.V. Gurjar, K. Masuda and M. Miyanishi, $\mathbb{A}^1$-fibrations on affine threefolds, J. pure and applied algebra **216** (2012), 296–313.

[9] R.V. Gurjar, K. Masuda and M. Miyanishi, Surjective derivations in small dimensions, preprint, to appear in the commemorative volume to the eightieth birthday of C.S. Seshadri.
[10] R.V. Gurjar, K. Masuda, M. Miyanishi and P. Russell, Affine lines on affine surfaces and the Makar-Limanov invariant, Canad. J. Math. **60** (1), 2008, 109–139.
[11] R. Hartshorne, Algebraic geometry, Springer, 1977.
[12] S. Iitaka, Deformations of compact complex surfaces, Global Analysis (Papers in Honor of K. Kodaira), 267–272, Univ. Tokyo Press, Tokyo, 1969.
[13] T. Kambayashi and M. Miyanishi, On flat fibrations by the affine line, Illinois J. Math. **22** (1978), 662–671.
[14] T. Kambayashi and D. Wright, Flat families of affine lines are affine line bundles, Illinois J. Math. **29** (1985), 672–681.
[15] Sh. Kaliman and M. Zaidenberg, Families of affine planes: the existence of a cylinder, Michigan Math. J. **49** (2001), no. 2, 353–367.
[16] Y. Kawamata, On deformations of compactifiable complex manifolds, Math. Ann. **235** (1978), no. 3, 247–265.
[17] Y. Kawamata, On the classification of non-complete algebraic surfaces, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), 215–232, Lecture Notes in Math., **732**, Springer, Berlin, 1979.
[18] K. Kodaira, On the stability of compact submanifolds of complex mnaifolds, Amer. J. Math. **85** (1963), 79–94.
[19] K. Kodaira, Complex manifolds and deformation of complex structures, Classics in Mathematics, Springer, 1986.
[20] J. Kollar, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer-Verlag, Berlin, 1996.
[21] J. Kollar and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998.
[22] M. Miyanishi, An algebro-topological characterization of the affine space of dimension three, Amer. J. Math. **106** (1984), 1469-1486.
[23] M. Miyanishi, Affine pseudo-coverings of algebraic surfaces, J. Algebra **294** (2005), 156–176.
[24] J.A. Morrow, Minimal normal compactifications of $\mathbb{C}^2$, Complex Analysis, 1972 (Proc. Conf. Rice Univ., Houston, Tex., 1972), Vol. I: Geometry of singularities, Rice Univ. Studies **59** (1973), no. 1, 97–112.
[25] W.D. Neumann, On the topology of curves in complex surfaces, Topological methods in algebraic transformation groups (New Brunswick, NJ, 1988), 117–133, Progr. Math., **80**, Birkhauser Boston, Boston, MA, 1989.
[26] C.P. Ramanujam, A topological characterization of the affine plane as an algebraic variety, Ann. of Math. **94** (1971), 69–88.
[27] P. Russell, Embeddings problems in affine algebraic geometry, Polynomial automorphisms and related topics, 113–135, Publishing House for Science and Technology, Hanoi, 2007.
[28] A. Sathaye, Polynomial ring in two variables over a DVR: a criterion, Invent. Math. **74** (1983), no. 1, 159–168.
[29] R.O. Wells, Differential Analysis on Complex Manifolds, Graduate Texts in Mathematics no. 65, Springer.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 001, India
E-mail address: gurjar@math.tifr.res.in

School of Science & Technology, Kwansei Gakuin University, 2-1 Gakuen, Sanda 669-1337, Japan
E-mail address: kayo@kwansei.ac.jp

Research Center for Mathematical Sciences, Kwansei Gakuin University, 2-1 Gakuen, Sanda 669-1337, Japan
E-mail address: miyanisi@kwansei.ac.jp