Open algebraic surfaces with finite group actions

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Abstract

Suppose that a finite group G acts algebraically on a log projective surface $(\overline{V}, \overline{\Delta})$ in such a way that $\overline{\Delta}$ is G-stable. Suppose $\kappa(\overline{V} - \overline{\Delta} \cup \operatorname{Sing} \overline{V}) = -\infty$. We first look for the conditions under which the Mori theory can be applied equivariantly and observe what we obtain from the Mori theory. We then consider the $\mathbb{P}^1$-fibrations which are preserved by the actions of G.

0 Introduction

Let G be a finite group. Consider the set of log projective surfaces $(\overline{V}, \overline{\Delta})$ defined over a fixed, algebraically closed, gound field k of characteristic zero which admit effective algebraic G-actions. We say that a morphism $f : (\overline{V}, \overline{\Delta}) \to (\overline{W}, \overline{\Gamma})$ is a *G-morphism* (or *G-equivariant morphism*) if f commutes with the G-actions. We can define the notion of relatively minimal (or minimal) model with respect to the birational G-morphisms.

The objective of the present article is to consider the equivariant classification of such G-relatively minimal log projective surfaces in the case where the log Kodaira dimension of $(\overline{V}, \overline{\Delta})$ is $-\infty$, i.e., $\overline{\kappa}(\overline{V} - \overline{\Delta} \cup \operatorname{Sing} \overline{V}) = -\infty$.

As our attempt, we consider first in section one the G-invariant cone theorem (cf. Lemma 1.1) under some technical hypotheses which enable us to make use of the cone theorem for the quotient surface $\overline{V}//G$ and to pull it back to the G-invariant setting on $\overline{V}$. We consider the case where $\overline{V}//G$ has Picard rank not less than two. Then we find a G-invariant extremal ray $\overline{F}$ such that $(\overline{F}^2) \le 0$, where G acts transitively on the set of irreducible components of $\overline{F}$. Let (V, D) be the minimal resolution of $(\overline{V}, \overline{\Delta})$, where D is the sum of the proper transform of $\overline{\Delta}$ on V and the exceptional components.

Let F be the proper transform of $\overline{F}$ on V. If $(\overline{F}^2) < 0$, F consists of mutually disjoint (-1) curves and can be contracted in a G-equivariant way. Accordingly, one can contract $\overline{F}$ on $\overline{V}$ in a G-equivariant way to produce a log projective surface with a G-action. This observation leads us to the notion of relatively minimal model in the G-equivariant setting. Suppose that $(\overline{F}^2) = 0$. Then there exists a $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to B$ such that a multiple of $\overline{F}$ is algebraically equivalent to the sum of fibers of $\overline{\rho}$ and that the translates of a fiber of $\overline{\rho}$ under the G-action are also the fibers of $\overline{\rho}$, i.e., G *preserves the fibration* $\overline{\rho}$. If $(F^2) < 0$, then there are singular points of $\overline{V}$ lying on $\overline{F}$. The possible types of the singular points on $\overline{F}$ are determined under some hypotheses on G and its action on $\overline{F}$ (cf. Lemma 1.4). If $(F^2) = 0$, then there are no singular points lying on $\overline{F}$. It is shown in Lemma 1.6 that F consists of a disjoint union of the fibers of $\overline{\rho}$, each of which consists of two (-1) curves $C + C'$ with $C' = C^g$ for some $g \in G$.

In section two, we consider the $\mathbb{P}^1$-fibration $\rho : V \to B$ which is induced on the minimal resolution (V, D) of $(\overline{V}, \overline{\Delta})$ by the $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to B$. The fibration ρ is hence preserved by the G-action. We first prove that ρ is necessarily a $\mathbb{P}^1$-bundle provided every fiber of ρ is either irreducible or consists of two (-1) curves $C + C'$ with $C' = C^g$ for some $g \in G$ and the divisor D contains a horizontal component (cf. Theorem 2.2). If we look at a fiber of ρ consisting of two (-1) curves $C + C'$ with $C' = C^g$ for $g \in G$, it is shown by a local argument near the fiber that g is an involution (cf. Theorem 2.6).

In section three, we treat a related question of bounding the number of fixed points under a nontrivial automorphism of a given smooth polarized projective variety (cf. the question at the beginning of Section 3). We have only a result in the case of dimension one due to M. Namba.

We are motivated by Zhang [9] to consider this kind of equivariant classification. In the case $\overline{V}$ is a smooth rational surface and $\overline{\Delta} = 0$, the results treated in the present article are, in fact, contained in [9].

1 Mori theory in the equivariant settings

Let $(\overline{V}, \overline{\Delta})$ be a log projective surface (cf. [7]). Namely, $\overline{V}$ is a normal projective surface and $\overline{\Delta}$ is a reduced effective Weil divisor such that $(\overline{V}, \overline{\Delta})$ has log terminal singularities. For log terminal singularities, see also [3] and [5]. In particular, the singularities of $\overline{V}$ lying on $\overline{\Delta}$ are cyclic quotient singularities whose resolution graph has one (and only one) terminal component meeting the proper transform of $\overline{\Delta}$ transversally in one point.Suppose that a finite group G acts algebraically on $\overline{V}$ in such a way that $\overline{\Delta}$ is G-stable. Let $\overline{W}$ be the algebraic quotient $\overline{V}//G$ and let $\pi : \overline{V} \to \overline{W}$ be the quotient morphism. Set $\overline{\Gamma} = \pi_*(\overline{\Delta})$. Then $\overline{W}$ is a normal projective surface with at worst quotient singularities. Let $f : (V, D) \to (\overline{V}, \overline{\Delta})$ be the minimal resolution and let $X = \overline{V} - \overline{\Delta} \cup \operatorname{Sing} \overline{V}$, where D is the sum of the proper transform of $\overline{\Delta}$ and the exceptional components of f.

We note that $f^*(\overline{\Delta} + K_{\overline{V}})$ is written as $D^\# + K_V$, where $D^\#$ is an effective $\mathbb{Q}$-divisor supported by the irreducible components of D. In fact, if $D = \sum_{i=1}^n C_i$ is the irreducible decomposition of D, then $D^\# = \sum_{i=1}^n \alpha_i C_i$ with $0 \leq \alpha_i \leq 1$. The integral part $[D^\#]$ is, in general, contained in the proper transform of $\overline{\Delta}$ and $D^\# - [D^\#]$ is denoted by $\operatorname{Bk} D$. Given a reduced effective divisor D on V, one can find the $\mathbb{Q}$-divisor $D^\#$ by peeling all possible rational admissible maximal twigs, rods and forks. For the details on the theory of peeling, see [7, p. 94].

We assume that the following conditions hold:

(1) $(\overline{W}, \overline{\Gamma})$ *is a log projective surface.*

(2) $\overline{\kappa}(V - D) = -\infty$ *and* $[D^\#] = \Delta$*, where* Δ *is the proper transform of* $\overline{\Delta}$ *on* V*. The second condition implies that all possible rational admissible maximal twigs (resp. rods, forks) of* D *are peeled off.*

By applying the Mori theory to the quotient surface $\overline{W}$, we have

$$\overline{\mathrm{NE}}\,(\overline{W}) = \overline{\mathrm{NE}}_{\,(\overline{\Gamma}+K_{\overline{W}})\geq 0}(\overline{W}) + \sum_{\overline{F}\in\mathcal{F}'} \mathbb{R}_+ \overline{F},$$

where $\mathcal{F}'$ is a countable set of the extremal curves $\overline{F}$ on $\overline{W}$ with $\overline{F}$ being represented by an irreducible curve $\overline{C}$ on $\overline{V}$ and where

$$\overline{\mathrm{NE}}_{\,(\overline{\Gamma}+K_{\overline{W}})\geq 0}(\overline{W}) = \{\eta \in \overline{\mathrm{NE}}\,(\overline{W}) \mid (\eta \cdot \overline{\Gamma} + K_{\overline{W}}) \geq 0\}.$$

Since $\pi^*(\overline{\mathrm{NE}}\,(\overline{W})) = \overline{\mathrm{NE}}\,(\overline{V})^G$, we have

$$\overline{\mathrm{NE}}\,(\overline{V})^G = \overline{\mathrm{NE}}_{\,\pi^*(\overline{\Gamma}+K_{\overline{W}})\geq 0}(\overline{V})^G + \sum_{\overline{C}\in\mathcal{F}'} (\sum_{g\in G} \overline{C}^g)\mathbb{R}_+,$$

where

$$\overline{\mathrm{NE}}_{\,\pi^*(\overline{\Gamma}+K_{\overline{W}})\geq 0}(\overline{V})^G = \{\xi \in \overline{\mathrm{NE}}\,(\overline{V})^G \mid (\xi \cdot \pi^*(\overline{\Gamma} + K_{\overline{W}})) \geq 0\}.$$

Note that

$$\overline{\Delta} + K_{\overline{V}} = \pi^*(\overline{\Gamma} + K_{\overline{W}}) + \overline{R},$$

where $\overline{R}$ is the log ramification divisor which is an effective divisor (cf. Iitaka [2]). We assume, furthermore, that

(3) *$\overline{R}$ is a nef divisor on $\overline{V}$.*

Then it is clear that

$$\begin{aligned} \overline{\mathrm{NE}}_{\pi^*(\overline{\Gamma}+K_{\overline{W}})\geq 0}(\overline{V})^G &\subseteq \overline{\mathrm{NE}}_{(\overline{\Delta}+K_{\overline{V}})\geq 0}(\overline{V})^G \\ &:= \{\xi \in \overline{\mathrm{NE}}\,(\overline{V})^G \mid (\xi \cdot \overline{\Gamma} + K_{\overline{W}}) \geq 0\}. \end{aligned}$$

Hence we obtain the following result.

Lemma 1.1 *With the above notations and assumptions, we have*

$$\overline{\mathrm{NE}}\,(\overline{V})^G = \overline{\mathrm{NE}}_{(\overline{\Delta}+K_{\overline{V}})\geq 0}(\overline{V})^G + \sum_{\overline{C}\in\mathcal{F}} (\sum_{g\in G} \overline{C}^g)\mathbb{R}_+,$$

where $\mathcal{F}$ is a countable subset of $\mathcal{F}'$.

Remarks (1) The condition (1) does not always hold as shown by the following example. Let (x, y) be the coordinates of the affine plane $\mathbb{A}^2$. Let G be a cyclic group of order n which is identified with the group of the n-th roots of unity $\{\zeta^i \mid 0 \leq i < n\}$ in the field k. Let G act on $\mathbb{A}^2$ by $\zeta(x, y) \mapsto (\zeta x, \zeta^d y)$ with $0 < d < n$. Embed $\mathbb{A}^2$ into $\mathbb{P}^2$ by $(x, y) \mapsto (x, y, 1)$ and extend the G-action as $\zeta(x, y, 1) \mapsto (\zeta x, \zeta^d y, 1)$. Set $V = \mathbb{P}^2$ and $\Delta = \ell_y + \ell_x$, where ℓ_y (resp. ℓ_x) are the lines on $\mathbb{P}^2$ defined by $y = 0$ (resp. $x = 0$). Set $\overline{W} = V//G$ and let$\overline{\Gamma}$ be the image of Δ on $\overline{W}$. Then $(\overline{W}, \overline{\Gamma})$ is not a log projective surface, for the images $\overline{\ell}_y, \overline{\ell}_x$ of ℓ_y, ℓ_x meet in a point where $\overline{W}$ has cyclic quotient singularity.

(2) The condition (3) that the log ramification divisor $\overline{R}$ is nef does not always hold, either. In the above example, blow up the point of origin $(0, 0)$ to obtain the Hirzebruch surface F_1. Suppose $d = 1$. Then the group G acts trivially on the minimal section M of F_1. Set $V = F_1$ and $\Delta = \ell_1 + \ell_2$ with the respective proper transforms ℓ_1, ℓ_2 of X, Y. Set $\overline{W} = V//G$ and let $\overline{\Gamma}$ be the image of Δ on $\overline{W}$. Then $\overline{W}$ is the Hirzebruch surface F_n and the image $\overline{M}$ of M is the minimal section. Let $\overline{\Gamma}$ be the image of Δ on $\overline{W}$. A straightforward computation shows that the log ramification divisor $\overline{R}$ in

the condition (3) is linearly equivalent to $2M + \ell$, where ℓ is a fiber of the $\mathbb{P}^1$-fibration on F_1. Then $\overline{R}$ is not nef, for $(\overline{R} \cdot M) = -1 < 0$.

Let $\overline{F} = (\sum_{g \in G} \overline{C}^g)/|H|$ be a G-invariant extremal curve, where H is the isotropy group of an irreducible component $\overline{C}$, i.e., $H = \{g \in G \mid \overline{C}^g = \overline{C}\}$. *We first assume that* $\rho(\overline{V}//G) \geq 2$. The case $\rho(\overline{V}//G) = 1$ will be treated later. Then, by the contraction theorem of Kawamata [4], there exists a nef divisor $\overline{H}$ on $\overline{V}$ such that $(\overline{H} \cdot \overline{F}) = 0$. Hence $(\overline{F}^2) \leq 0$ by the Hodge index theorem. To go further, we note that $D^\# + K_V = f^*(\overline{\Delta} + K_{\overline{V}})$ and $\mathrm{Bk}\, D$ are G-stable $\mathbb{Q}$-divisors and that $\mathrm{Sing}\, \overline{V}$ is a G-stable set.

Case 1. Suppose $(\overline{F}^2) < 0$. Since $(\overline{F} \cdot \overline{\Delta} + K_{\overline{V}}) < 0$, we have $(F \cdot D^\# + K_V) < 0$, where $F = \sum_{g \in G} C^g/|H|$ is the proper transform of $\overline{F}$. Furthermore, $(F^2) < 0$ because $f^*(\overline{F}) = F + Z$ for an effective $\mathbb{Q}$-divisor Z with $\mathrm{Supp}\, Z \subseteq \mathrm{Supp}\, \mathrm{Bk}\, D$. Hence, for the component C of F, it follows that $(C \cdot D^\# + K_V) < 0$ and $C + \mathrm{Bk}\, D$ is negative definite, for $(C \cdot D^\# + K_V) = (C^g \cdot D^\# + K_V)$ for every $g \in G$ and $(\overline{C}^2) < 0$. We have the following result.

Lemma 1.2 *If* $(F^2) < 0$ *and* C *is not* G*-stable, then* $C \cap C^g = \emptyset$ *whenever* $C^g \neq C$ *for* $g \in G$.

Proof. Let H be the isotropy subgroup of C, and let $\{g_1, \ldots, g_s\}$ be a set of representatives of the right coset decomposition G/H. Let $g_1 = e$ the identity and let $C_i = C^{g_i}$. Suppose $C \neq C^g$ and $(C \cdot C^g) > 0$. If $C \not\subset \Delta$, then C_i is a (-1) curve. Furthermore, each C_i has at least one C_j such that $(C_i \cdot C_j) > 0$. Hence we have

$$\begin{aligned} (F^2) &= ((C_1 + \cdots + C_s)^2) \\ &= \sum_{i=1}^{s} (C_i \cdot C_1 + \cdots + C_s) \\ &\geq (-1+1) + \cdots + (-1+1) = 0, \end{aligned}$$

which is a contradiction. Suppose $C \subset \Delta$. Since

$$0 > (C \cdot D^\# + K_V) \geq (C \cdot C + K_V),$$

it follows that $C \cong \mathbb{P}^1$ and $(C^2) < 0$, for $C + \mathrm{Bk}\, D$ is negative definite.

Suppose $(C^2) \leq -2$. Since $C^g \subset \Delta$ for $g \in G$ and $(C \cdot D^\# + K_V) < 0$, it follows that $(C \cdot C^g) = 0$ or 1 and $(C \cdot C^g) = 1$ possibly for only one translate

C^g. Furthermore, we have

$$(C \cdot D^{\#} - \sum_{i=1}^{s} C_i) < \begin{cases} 1 & \text{if} \quad C \cap C^g \neq \emptyset \quad \text{for some} \quad g \in \{g_2, \ldots, g_s\} \\ \\ 2 & \text{if} \quad C \cap C^g = \emptyset \quad \text{for every} \quad g \in \{g_2, \ldots, g_s\}. \end{cases}$$

Let $B_1, \ldots, B_t$ be all the irreducible components of $\mathrm{Bk}\, D$ such that $(C \cdot B_i) > 0$. Then we have

$$\sum_{i=1}^{t} \left(1 - \frac{1}{b_i}\right) < 1 \quad (\text{or} \quad 2),$$

where $b_i = -({B_i}^2)$. This implies that $t \leq 1$ in the case $C \cap C^g \neq \emptyset$ and $t \leq 3$ if $C \cap C^g = \emptyset$ for every $g \in \{g_2, \ldots, g_s\}$. In the case $t \leq 1$ or in the case $C \cap C^g = \emptyset$ for every $g \in \{g_2, \ldots, g_s\}$ and $t = 2$, the connected component of $F + \mathrm{Bk}\, D$ containing C is an admissible rational rod. This contradicts the hypothesis that $\Delta = [D^{\#}]$. Hence C is a (-1) curve. In the case $C \cap C^g = \emptyset$ for every $g \in \{g_2, \ldots, g_s\}$ and $t = 3$, C has three twigs sprouting from it. Let d_1, d_2, d_3 be the absolute values of the discriminants of the intersection forms of the three twigs. Then the condition $(C \cdot D^{\#} - C) < 2$ is stated as

$$\sum_{i=1}^{3} \left(1 - \frac{1}{d_i}\right) < 2.$$

Hence $\{d_1, d_2, d_3\}$ is, up to permutations, one of the Platonic triplets. Hence the connected component of D containing C is an admissible rational fork. This contradicts again the condition $\Delta = [D^{\#}]$. Hence C is a (-1) curve. The rest of the proof is the same as in the previous case $C \not\subset \Delta$. Q.E.D.

By Lemma 1.2, we can contract $\{C^g \mid g \in G\}$ simultaneously without losing the G-action if $(\overline{F}^2) < 0$. Since the condition $(\overline{F}^2) < 0$ implies that $F + \mathrm{Bk}\, D$ is negative definite, this implies that the contraction of $\overline{F}$ on $\overline{V}$ produces again a log projective surface with a G-action (cf. [7]). We say that a log projective surface $(\overline{V}, \overline{\Delta})$ with an action of a finite group G is G-*relatively minimal* if there is no curve $\overline{F} = \sum_{g \in G} \overline{C}^g / |H|$ such that $(\overline{F}^2) < 0$ and $(\overline{F} \cdot \overline{\Delta} + K_{\overline{V}}) < 0$. In terms of a minimal resolution $f : (V, D) \to (\overline{V}, \overline{\Delta})$, it is equivalent to saying that there is no curve $F = \sum_{g \in G} C^g / |H|$ on V such that $(F^2) < 0$, $F + \mathrm{Bk}\, D$ is negative definite and $(C \cdot D^{\#} + K_V) < 0$.

Given an extremal curve $\overline{F} = \sum_{g \in G} \overline{C}^g$, we may assume, without loss of generality, that $(\overline{F}^2) = 0$, i.e., $F + \mathrm{Bk}\, D$ is not negative definite, but negative semi-definite.

CASE 2. Suppose $(\overline{F}^2) = 0$. Write $F = \sum_{i=1}^{s} C_i$, which is the integral part of $f^*(\overline{F})$. Consider first the case $(F^2) < 0$. We leave the case $(F^2) = 0$ below. We then have the following result.

Lemma 1.3 *Suppose that $(\overline{F}^2) = 0$ and $(F^2) < 0$. Then the following assertions hold.*

(1) *If $(\overline{C}^2) < 0$, then the C^g with $g \in G$ are the mutually disjoint (-1) curves, and $C + \mathrm{Bk} D$ is negative definite, while $F + \mathrm{Bk} D$ is not negative definite. Furthermore, C (hence every C^g as well) is not a component of Δ.*

(2) *If $(\overline{C}^2) = 0$, then $(\overline{C} \cdot \overline{C}^g) = 0$ for every $g \in G$.*

(3) *There exists a $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to B$ such that a multiple of $\overline{F}$ is algebraically equivalent to the sum of fibers of $\overline{\rho}$ and that G preserves $\overline{\rho}$.* Here we say that G preserves $\overline{\rho}$ if the g-translate of a fiber of $\overline{\rho}$ is again a fiber of $\overline{\rho}$ for every $g \in G$.

Proof. (1) If $(\overline{C}^2) < 0$, then $C + \mathrm{Bk}\, D$ is negative definite. So, if $(F^2) < 0$, the argument in CASE 1 works in this case as well. Hence the C^g with $g \in G$ are the mutually disjoint (-1) curves. We can contract the C^g simultaneously, though some components of $\mathrm{Bk}\, D$ may not be contracted because $F + \mathrm{Bk}\, D$ is not negative definite. It is then clear that F has two or more irreducible components. Suppose $C \subset \Delta$ (hence every $C^g \subset \Delta$). Since $(\overline{V}, \overline{\Delta})$ is a log projective surface, for each $g \in G$, the connected component of $F + \mathrm{Bk}\, D$ containing C^g is an admissible rational rod of $\mathrm{Bk}\, D$ with one end component meeting C^g in one point transversally. Then $F + \mathrm{Bk}\, D$ is negative definite, which is not the case. So, C^g is not a component of Δ.

(2) Since $(\overline{F}^2) = (\overline{C}^2) = 0$, it is clear that $(\overline{C} \cdot \overline{C}^g) = 0$ for every $g \in G$.

(3) We note first that the proper transform $C = f'(\overline{C})$ is a smooth rational curve. In fact, if $(\overline{C}^2) < 0$, the assertion (1) shows that $f^*(\overline{F})$ consists of smooth rational curves. Suppose $(\overline{C}^2) = 0$. Since $(\overline{C} \cdot \overline{\Delta} + K_{\overline{V}}) = (C \cdot D^{\#} + K_V) < 0$, it follows that C is a smooth rational curve. Note that V is a ruled surface because $\overline{\kappa}(X) = -\infty$. If the irregularity q is positive, the curves in a connected component of $F + \mathrm{Bk}\, D$ are mapped to the same point by the Albanese morphism. Let $\overline{\rho} : \overline{V} \to B$ be the $\mathbb{P}^1$-fibration induced by the Albanese morphism of V. Then each connected component $\overline{L}$ of $\overline{F}$ is contained in a fiber of $\overline{\rho}$. Since $(\overline{F}^2) = 0$, a multiple of $\overline{L}$ is a fiber of $\overline{\rho}$.

Let ℓ be a general fiber of $\overline{\rho}$. Then ℓ is algebraically equivalent to $N\overline{L}$ with $N > 0$. Hence $\ell^g \approx N\overline{L}^g$. Since $(\overline{L} \cdot \overline{L}^g) = 0$ because $\overline{L}^g$ is also a connected component of $\overline{F}$, we have $(\ell \cdot \ell^g) = 0$. This implies that G preserves the fibration $\overline{\rho}$.

Now suppose that V is rational. Let n be a positive integer. By the Riemann-Roch theorem, we have

$$\begin{aligned} h^0(nf^*(\overline{F})) - h^1(nf^*(\overline{F})) &= \frac{1}{2}(nf^*(\overline{F}) \cdot nf^*(\overline{F}) - K_V) + 1 \\ &= -\frac{n}{2}(f^*(\overline{F}) \cdot K_V) + 1 \\ &> \frac{n}{2}(f^*(\overline{F}) \cdot D^{\#}) + 1 \end{aligned}$$

because

$$(f^*(\overline{F}) \cdot D^{\#} + K_V) = (\overline{F} \cdot \overline{\Delta} + K_{\overline{V}}) < 0.$$

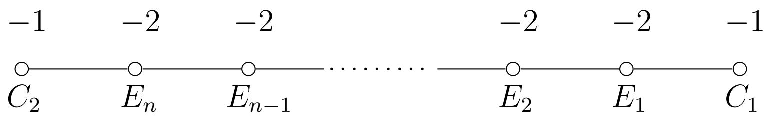
Suppose $(f^*(\overline{F}) \cdot D^{\#}) \geq 0$. Then $h^0(nf^*(\overline{F})) > 1$ for $n \geq 1$. So, $|n(f^*(\overline{F}))|$ defines a morphism $\rho : V \to B$ whose general fibers are irreducible. Since $(f^*(\overline{F}) \cdot K_V) < -(f^*(\overline{F}) \cdot D^{\#}) \leq 0$, it follows that ρ is a $\mathbb{P}^1$-fibration. Furthermore, ρ induces a $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to B$ such that $\rho = \overline{\rho} \cdot f$ and that a multiple of $f^*(\overline{F})$ consists of fibers. Since $\overline{F}$ is G-stable, we know that G preserves the fibration $\overline{\rho}$. Suppose that $(f^*(\overline{F}) \cdot D^{\#}) < 0$. Then $(\overline{F} \cdot \overline{\Delta}) < 0$. Hence each connected component $\overline{L}$ of $\overline{F}$ is a component of $\overline{\Delta}$ with $(\overline{L}^2) < 0$. This case does not occur by the hypothesis $(\overline{F}^2) = 0$. Q.E.D.

The reducible fiber of the $\mathbb{P}^1$-fibration in Lemma 1.3 which is supported by a connected component of $\overline{F}$ can be specified as follows under some additional hypotheses.

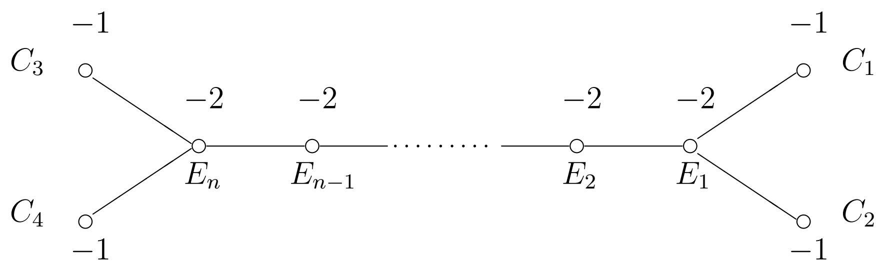
Lemma 1.4 *Let Φ be a reduced, reducible fiber of the $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to B$ in Lemma 1.3 whose support is a connected component $\overline{L}$ of $\overline{F}$. Write the proper transform $L := f'(\overline{L})$ on the minimal resolution V of $\overline{V}$ as $L = \sum_{i=1}^{s} C_i$. Then every C_i is a (-1) curve. Let G_0 be the subgroup of G consisting of elements g with $\overline{L}^g = \overline{L}$. Furthermore, if G_0 is an abelian group and G_0 acts on Φ effectively, the possible configurations of $f^*(\Phi)$ are then exhausted by the following list:*

(1) *$s = 2$, $G_0 \cong \mathbb{Z}/2\mathbb{Z}$ and the dual graph of $f^*(\Phi)$ is a linear graph which is the (-2) chain corresponding to the exceptional graph of a rational*

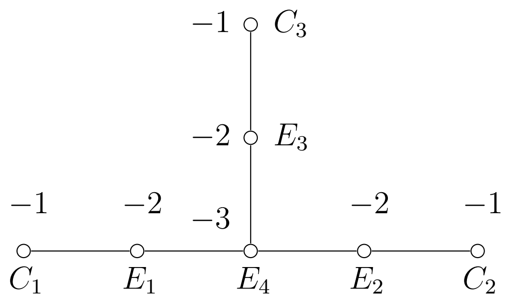
double point of type A_n with each of the both end components of the (-2) chain having a (-1) curve meeting it.



(2) *$s = 4$, $G_0 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the dual graph of $f^*(\Phi)$ is a linear graph which is the (-2) chain corresponding to the exceptional graph of a rational double point of type A_n with each of the both end components of the (-2) chain having two (-1) curves meeting it.*



(3) *$s = 3$, $G_0 \cong \mathbb{Z}/3\mathbb{Z}$ and the dual graph of $f^*(\Phi)$ is a (-3) component having three linear branches, each of which consists of a (-2) curve and a (-1) curve with the (-2) curve meeting the (-3) component.*



(4) *$s \geq 3$, the configuration of $f^*(\Phi)$ is a unique $(-s)$ curve E meeting s curves $C_1, \ldots, C_s$ which are (-1) curves, and G_0 is a finite subgroup of*

PGL $(2, k)$ *acting transitively on the set* $\{P_1, \ldots, P_s\}$ *of* $\mathbb{P}^1$ *with* $P_i = C_i \cap E$.

In particular, the fiber Φ *has a unique singular point.*

Proof. Note that $f^*(\Phi)$ is a degenerate fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface. Write

$$f^*(\Phi) = \sum_{i=1}^{s} aC_i + \sum_{j=1}^{n} b_j E_j,$$

where the coefficients of the C_i are the same becasuse every C_i is a translate of one of the C_i and where the B_j are the exceptional curves of the minimal resolution $f : V \to \overline{V}$. Here $s \geq 2$ and $a = 1$ because Φ is a reduced, reducible fiber by the hypothesis. Note that every C_i is a (-1) curve and an end component of $f^*(\Phi)$. This implies that every $\overline{C}_i := f(C_i)$ passes through one and only one singular point on Φ. Since this singular point is a quotient singular point, the exceptional graph is either a rod or a fork.

Suppose first that the exceptional graph $E := \sum_{j=1}^{n} E_j$ is a rod with $n \geq 2$. So, we may assume that $(E_i \cdot E_j) = 1$ (resp. 0) if $j = i + 1$ with $1 \leq i < n$ (resp. otherwise). Hence E_1 and E_n are the end components. Note that C_i meets E_1 or E_n, for otherwise the coefficient a must be larger than 1. Suppose $C := C_1$ meets E_1. We claim that the set H of elements $g \in G_0$ with $(C^g \cdot E_1) = 1$ is a subgroup of G_0. In fact, if $g, h \in H$, then $(C^g \cdot E_1) = (C^h \cdot E_1) = 1$. Suppose $(C^{gh} \cdot E_n) = 1$. Then h maps the point $C^g \cap E_1$ onto the point $C^{gh} \cap E_n$ on E_n. Hence h maps E_1 onto E_n. So, $(C^h \cdot E_1) = 0$. This is a contradiction. Indeed, H is a normal subgroup of G_0 of index 2. In particular, s is even. If $s = 2$, then the dual graph of $f^*(\Phi)$ is the case (1) listed above. So, assume that $s \geq 4$. We cliam that the stabilizer group of C is trivial. In fact, let $K = \{g \in H \mid C^g = C\}$. Then the stabilizer group of C_i is a conjugate of K. Since G_0 is abelian by the hypothesis, K acts on the curve E_1 and fixes at least three points $C_1 \cap E_1, C_i \cap E_1$ and $E_2 \cap E_1$, where $(C_i \cdot E_1) = 1$. So, K acts trivially on E_1. Consider the quotient space $V_1 := V//K$. Since G preserves the $\mathbb{P}^1$-fibration $\rho : V \to B$, there exists a $\mathbb{P}^1$-fibration $\rho_1 : V_1 \to B_1$. Then $C = \pi_1^*(\pi_1(C))$, where $\pi_1 : V \to V_1$ is the quotient morphism. This is so because $\pi_1|_C : C \to \pi_1(C)$ has degree $|K|$. Hence $(C^2) = |K|(\pi_1(C)^2)$. Since C is a (-1) curve, it follows that $|K| = 1$. Now the group H acts on E_1 so that the point $E_1 \cap E_2$ is a fixed point. Since H acts effectively on the affine line $E_1 - E_1 \cap E_2$, it follows that H is a cyclic group of order t, where $s = 2t$. In fact, write the H-action on $E_1 - E_1 \cap E_2$ as $h(t) = a(h)t + b(h)$ with $a(h) \in k^*$ and $b(h) \in k$, where t is a coordinate of the affine line $E_1 - E_1 \cap E_2$. Then $h \mapsto a(h)$ is a multiplicative character of H. We shall show that $a : H \to k^*$ is injective. Otherwise, take an element $h \neq 1$ from $\operatorname{Ker} a$. Then $h(t) = t + b(h)$ with $b(h) \neq 0$. Then h has infinite order, a contradiction. Hence a is injective, and H is a cyclic group. We assume that $C_i \cap E_1 \neq \emptyset$ for $1 \leq i \leq t$ and $C_i \cap E_n \neq \emptyset$ for $t+1 \leq i \leq s$. After the contraction of $C_1, \ldots, C_t$, the component E_1 becomes a (-1) curve. Hence $({E_1}^2) = -(t+1)$. Set $P := E_1 \cap E_2$. The action of H near the point P is given as $(x, y) \mapsto (\zeta x, \zeta^d y)$, where ζ is a primitive t-th root of unity, $\{x, y\}$ is a system of local parameters at P such that E_1 (resp. E_2) is defined by $y = 0$ (resp. $x = 0$) and d is an integer $0 \leq d < t$. Note that the action of H is thus normalized because H acts effectively on E_1. Consider the quotient space $V_2 := V//H$ with quotient morphism $\pi_2 : V \to V_2$. Let $\overline{E}_1$ and $\overline{E}_2$ be the images of E_1 and E_2 by π_2, respectively. The point $\overline{P} := \pi_2(P)$ is a cyclic quotient singular point of type (t, d) if $d > 0$ and a smooth point if $d = 0$. By Sublemma below, we have $({\overline{E}_1}^2) = ({E'_1}^2) + (d/t)$, where E'_1 is the proper transform of $\overline{E}_1$ on the minimal resolution of V_2. We then have

$$-(t+1) = ({E_1}^2) = t({\overline{E}_1}^2) = t\left(({E'_1}^2) + \frac{d}{t}\right).$$

Since $({E'_1}^2)$ is an integer, it follows that $d = t - 1$ and $({E'_1}^2) = -2$. Then we can cntract H-equivariantly the components $C_1, \ldots, C_t$ and E_1. On the minimal resolution of V_2, we contract the image of C_1 which is a (-1) curve, the component E'_1 and the linear chain of length $t-1$ of the exceptional (-2) curves arising from the resolution of singularity of the point $\overline{P}$. Repeating the above argument, we find that the configuration of the curves $\sum_{i=1}^{s} C_i + \sum_{j=1}^{n} E_j$ together with the H-actions are described as follows, where $P_i := E_i \cap E_{i+1}$ for $1 \leq i < n$ and ζ is a primitive t-th root of unity. The H-action near the point P_i is given by $(x_i, y_i) \mapsto (\zeta x_i, \zeta^{-1} y_i)$, where $\{x_i, y_i\}$ be a system of local parameters such that E_i (resp. E_{i+1}) is defined by $y_i = 0$ (resp. $x_i = 0$).

Sublemma *With the above notations, we have*

$$({\overline{E}_1}^2) = ({E'_1}^2) + \frac{d}{t},$$

Proof. Write t/d in the form of a continued fraction

$$\frac{t}{d} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_s}}}$$

Then the total transform of $\overline{E}_1 + \overline{E}_2$ in the minimal resolution of singularity at $\overline{P}$ is a linear chain

$$E_1' + G_1 + \cdots + G_s + E_2',$$

where $({G_i}^2) = -a_i$ for $1 \le i \le s$. Then the total transform of $\overline{E}_1$ in the minimal resolution is written as

$$E_1' + \alpha_1 G_1 + \alpha_2 G_2 + \cdots + \alpha_s G_s,$$

where $\alpha_1, \ldots, \alpha_s$ are determined by the conditions

$$\begin{gathered} 1 - a_1\alpha_1 + \alpha_2 = 0 \\ \alpha_1 - a_2\alpha_2 + \alpha_3 = 0 \\ \\ \alpha_{s-2} - a_{s-1}\alpha_{s-1} + \alpha_s = 0 \\ \alpha_{s-1} - a_s\alpha_s = 0 \end{gathered}$$

Then it is easy to verify that

$$({\overline{E}_1}^2) = ({E_1'}^2) + \alpha_1 = ({E_1'}^2) + \frac{d}{t}.$$

Q.E.D.

Case $n = 2m+1$. The H-action stabilizes each of the components E_j $(1 \le j \le n)$, and G_0/H flips the two branches $\sum_{i=1}^t C_i + \sum_{j=1}^m E_j$ and $\sum_{i=t+1}^s C_i + \sum_{j=m+2}^n E_j$. Let σ be an element not in H. Then $\sigma(E_m) = E_{m+2}$. Since $|G_0/H| = 2$, $\sigma^2 \in H$. So, write $\sigma^2 = \tau^r$, where $\tau \in H$ is a generator of H and $0 \le r < t$. Since σ is an automorphism of finite order of $E_{m+1} \cong \mathbb{P}^1$, it fixes two points, say Q_1, Q_2, and acts on $\mathbb{A}^1_* := \mathbb{P}^1 - \{Q_1, Q_2\}$ as an element of G_m. So, if $r > 0$, then the points Q_1, Q_2 coincide with P_m, P_{m+1} up to a permutation. This is a contradiction because $\sigma(P_m) = P_{m+1}$. Hence

$\sigma^2 = 1$, and $\{Q_1, Q_2\} \cap \{P_m, P_{m+1}\} = \emptyset$. Note that σ is an involution of the component E_{m+1} commuting with the H-action. Write $E_{m+1} - \{P_m, P_{m+1}\} =$ Spec $k[x, x^{-1}]$, where P_m and P_{m+1} are respectively defined by $x = 0$ and $x = \infty$. Choose the coordinate x so that Q_1 is defined by $x = 1$ and choose, furthermore, a generator τ of H such that $\tau(x) = \zeta x$. Since σ acts on an automorphism of $k[x, x^{-1}]$, we have $\sigma(x) = \alpha x^{-1}$ with $\alpha \in k^*$ because $\sigma(P_m) = P_{m+1}$. Since $\sigma(Q_1) = Q_1$ and Q_1 is defined by $x = 1$, it follows that $\alpha = 1$. So, $\sigma(x) = x^{-1}$. Then it is easy to show that $\sigma\tau\sigma = \tau^{-1}$. Thus G_0 is a dihedral group of order $2t$, which is not abelian unless $t = 2$. If $t = 2$, then the graph of Φ is the case (2).

Case $n = 2m$. Let σ be an element of G_0 not in H. Then G_0 is generated by σ and $H = \langle \tau \rangle$. Note that H fixes the points P_i $(1 \le i \le n-1)$ and σ fixes the mid-point P_m. Then $\sum_{i=1}^{s} C_i + \sum_{j \ne m, m+1} E_j$ is stabilized by G_0. Hence we can contract this divisor G_0-equivariantly. After the contraction, we obtain a smooth projective surface V_3 with a $\mathbb{P}^1$-fibration $\rho_3 : V_3 \to B$ on which the group G preserves the fibration ρ_3, i.e., ρ_3 maps the fibers of ρ_3 onto the fibers of ρ_3. The image of the fiber Φ consists of two (-1) curves $\overline{E}_m$ and $\overline{E}_{m+1}$ meeting transversally in the point $\overline{P}_m$. The element σ acts on the fiber in such a way that $\sigma(\overline{E}_m) = \overline{E}_{m+1}$. Then, by Theorem 2.6 below whose proof is independent of the present argument, σ is an involution. Tracing the images of a point P chosen on $E_m - \{P_m, P_{m-1}\}$ by the automorphisms σ and τ, we know that $\sigma\tau\sigma = \tau^{-1}$. In fact, choose a coordinate x on E_m so that P_m is defined by $x = 0$ and $\tau(x) = \zeta x$. Then we can take $x' = \sigma(x)$ as a coordinate on E_{m+1} such that P_m is defined by $x' = 0$ and $\tau(x') = \zeta^{-1} x'$. So, we obtain the same result as in the case $n = 2m + 1$.

Suppose next that the exceptional graph E is a unique irreducible component. Then the curve C_i intersects E transversally in a point P_i. We may assume that $s \ge 3$. Then $K = \{g \in G_0 \mid C^g = C\}$ is trivial and the restriction $G_0 \to \mathrm{Aut}\,(E)$ is injective. Hence G is a finite subgroup of PGL $(2, k)$ acting on the set $\{P_1, \ldots, P_s\}$ transitively. So, the graph of Φ is the case (4).

Finally, consider the case the exceptional graph E is a fork. Since a fork has a (-2) component meeting the central component E_4, we denote the (-2) component by E_1. There is only one (-1) component C_1 meeting E_1. In fact, there is at most one (-1) component meeting E_1. If there are none of them, then E_1 must remain after the contraction of all the (-1)

components and all subsequently contractible components of $f^*(\Phi)$. But this is not the case. On the other hand, if $g \in G$ maps C_1 to C_i, then g maps E_1 to a (-2) component of the exceptional graph which is adjacent to C_i. This observation shows that the curve C_i and a (-2) curve E_i form a linear branch connected to the component E_4. The graph of $f^*(\Phi)$ is thus the case (3) in the list. The group G_0 acts on the component E_4 and permutes three points $P_i := E_i \cap E_4$ for $i = 1, 2, 3$. Suppose that G_0 contains an element τ which permutes cyclically three points P_1, P_2, P_3. Then $G_0 \cong \mathbb{Z}/3\mathbb{Z}$ as long as G_0 is abelian. Suppose that any element of G_0 fixes at least one point of P_1, P_2, P_3. Then there exist σ_1, σ_2 such that $\sigma_1(P_1) = P_2, \sigma_2(P_1) = P_3$. Then $\sigma_1\sigma_2$ permutes three points cyclically. This is a contradiction. Hence $G_0 \cong \mathbb{Z}/3\mathbb{Z}$. Q.E.D.

The observation in CASE $n = 2m + 1$ in the proof of Lemma 1.4 implies the existence of a fiber Φ such that $f^*(\Phi)$ is a linear chain of (-2) curves with each of the end components of the chain having t of the (-1) curves meeting it if one admits the dihedral group. Here is an example.

EXAMPLE **1.5** *Let $\rho : W \to B$ be a $\mathbb{P}^1$-fibration on a log projective surface W. We assume that the following conditions are satisfied.*

(1) *Let H be a cyclic group of order t. The group H acts on W so that $\rho \cdot g = \rho$ for every $g \in G$. Hence, for every smooth fiber F of ρ, there are two points P_0, Q_0 such that P_0, Q_0 are the fixed points and H acts on $\mathbb{A}^1_* = F - \{P_0, Q_0\}$ via the natural G_m-action.*

(2) *Choose a smooth fiber F as above. Choose an inhomogeneous coordinate x on F such that P_0, Q_0 are defined respectively by $x = 0, x = \infty$ and that $\tau(x) = \zeta x$ for a generator τ of H, where ζ is a primitive t-th root of unity. Consider an involution σ on W such that σ acts on F as $\sigma(x) = x^{-1}$. Then $\sigma\tau\sigma = \tau^{-1}$ on F. Hence the subgroup G of $\mathrm{Aut}\,(W)$ generated by τ and σ is the dihedral group of order $2t$.*

Such an example of (W, p) with a G-action does exist. For example, consider the above G-action on $\mathbb{P}^1$ and take a direct product $\mathbb{P}^1 \times B$.

Blow up the points P_0, Q_0 and obtain the exceptional curves L_1, R_1, respectively. By the abuse of notations, we denote the proper transform of F by the same letter F and denote the intersection points $F \cap L_1$ and $F \cap R_1$ by the letters P_0 and Q_0, respectively. Then the group H acts on L_1, R_1. Near the point P_0 on L_1, choose a system of local parameters $\{x, y\}$ such that x

is as above and y is an inhomogeneous coodinate on L_1 with the point P_0 defined by $y = 0$. Then the action of H on L_1 is given by $\tau(y) = \zeta^{-1}y$. Hence there is another H-fixed point P_1 on L_1. Similarly, there is a H-fixed point Q_1 on R_1. Blow up the points P_1, Q_1 to obtain the exceptional curves L_2, R_2, respectively. Continue this process to obtain a linear chain

$$L_m + L_{m-1} + \cdots + L_1 + F + R_1 + \cdots + R_{m-1} + R_m,$$

where we can extend the G-action onto the blown-up surfaces in such a way that $\sigma(L_i) = R_i$ and $\sigma(R_i) = L_i$ for $1 \le i \le m$. Choose the points $A_1, \ldots, A_t$ on L_m which constitute the H-orbit of A_1. Let $B_j = \sigma(A_j)$ for $1 \le j \le t$. Now blow up these $2t$ points $A_1, \ldots, A_t$ and $B_1, \ldots, B_t$ to obtain the exceptional curves $C_1, \ldots, C_t$ and $C_{t+1}, \ldots, C_s$, respectively. Let V be the surface obtained by the above sequence of blowing-ups. Then the dihedral group G acts transitively on the set of (-1) curves $\{C_1, \ldots, C_t, C_{t+1}, \ldots, C_s\}$. The surface V has a $\mathbb{P}^1$-fibration $\rho : V \to B$ which extends the fibration p and contains $\sum_{i=1}^s C_i + \sum_{j=1}^m (L_j + R_j) + F$ as a fiber. The linear chain $\sum_{j=1}^m L_j + F + \sum_{j+1}^m R_j$ which consists of (-2) linear chains with two $-(t+1)$ curves attached to the end components contracts to a cyclic quotient singular point.

Case 3. Now suppose that $(\overline{F}^2) = (F^2) = 0$. It follows that $\overline{F} \cap \mathrm{Sing}\,\overline{V} = \emptyset$. Then we have the following result.

Lemma 1.6 *Suppose $s \ge 2$, i.e., $C = C_1$ is not G-stable. Then the following assertions hold.*

(1) *Suppose $(C_1 \cdot C_i) \neq 0$ for some $i \neq 1$. Then F is rewritten as*

$$F = \sum_{i=1}^{r} (C_i + C_i'),$$

where $s = 2r, ({C_i}^2) = ({C_i'}^2) = -1$ and $(C_i \cdot C_i') = 1$. Hence there exists a $\mathbb{P}^1$-fibration $\rho : V \to B$ such that the G-action preserves the $\mathbb{P}^1$- fibration.

(2) *Suppose $(C_1 \cdot C_i) = 0$ for every $i \neq 1$. Then $({C_i}^2) = 0$ for every i, and there exists a $\mathbb{P}^1$-fibration $\rho : V \to B$ such that the G-action preserves the $\mathbb{P}^1$-fibration.*

(3) *The* $\mathbb{P}^1$*-fibration* $\rho : V \to B$ *in (1) and (2) above factors as* $\rho = \overline{\rho} \cdot f$*, where* $\overline{\rho} : \overline{V} \to B$ *is a* $\mathbb{P}^1$*-fibration.*

Proof. (1) Since $({C_i}^2) < 0$ and $(C_i \cdot D^{\#} + K_V) < 0$, it follows that every C_j is a (-1) curve if some C_i is not contained in Δ and a smooth rational curve if C_i is contained in Δ. Set $({C_i}^2) = -a$ with $a \geq 1$ and $t = (C_1 \cdot \sum_{i=2}^{s} C_i)$. Since every C_j is a translate of C_1 by an element of G, we have $(C_j \cdot \sum_{i \neq j} C_i) = t$. Since $(F^2) = 0$, we have

$$\begin{aligned}(F^2) &= \sum_{i=1}^{s}(C_i \cdot \sum_{j=1}^{s} C_j) \\ &= \sum_{i=1}^{s}\left\{({C_i}^2) + (C_i \cdot \sum_{j \neq i} C_j)\right\} \\ &= -sa + st,\end{aligned}$$

which yields that $t = a$ because $(F^2) = 0$. We shall show that $a = 1$, i.e., every C_i is a (-1) curve. Suppose $a \geq 2$. Since $\overline{\kappa}(X) = -\infty$, we know that $|D + K_V| = \emptyset$. Suppose that V is rational. It then follows that D and hence Δ has a tree as the dual graph. Since $C_1 + \cdots + C_s$ is a part of Δ, its dual graph is a tree. So, some component C_i meets at most one component of $\sum_{j \neq i} C_j$. But the above remark shows that the number of the components in $\sum_{j \neq i} C_j$ that C_i meets is exactly $a \geq 2$. This is a contradiction. If V is irrational, V is a ruled surface because $|n(D + K_V)| = \emptyset$ for every $n > 0$. Since C_i is a rational curve, it is a fiber component. Since the dual graph of a degenerate fiber of a $\mathbb{P}^1$-fibration has a tree as the dual graph, the dual graph of $C_1 + \cdots + C_s$ is a tree. The above argument for the rational case works in the irrational case as well. Hence every C_i is a (-1) curve, and we may rewrite F as

$$F = \sum_{i=1}^{r}(C_i + C_i'),$$

where $({C_i}^2) = ({C_i'}^2) = -1$ and $(C_i \cdot C_i') = 1$. Since $((C_i + C_i')^2) = 0$ and $C_1 + C_1'$ is nef, a complete linear system $\Lambda = |N(C_1 + C_1')|$ with $N \gg 0$ defines a $\mathbb{P}^1$-fibration $\rho : V \to B$, where B is a smooth complete curve. Let ℓ be a general fiber of ρ. We shall show that ℓ^g is a fiber of ρ as well. In fact, note that $(C_1 + C_1' \cdot C_i + C_i') = 0$ for every $1 \leq i \leq r$ and the $C_i + C_i'$ $(1 \leq i \leq r)$

exhaust all the G-translates of $C_1 + C'_1$. Hence $C_i + C'_i$ is a fiber of ρ. Since

$$\begin{aligned}(\ell^g \cdot C_1 + C'_1) &= (\ell \cdot (C_1 + C'_1)^{g^{-1}}) \\ &= (\ell \cdot C_i + C'_i) = 0\end{aligned}$$

for some i, we know that ℓ^g is a fiber of ρ.

(2) If $(C_1 \cdot C_i) = 0$ for every $i \neq 1$, then $(C_i^2) = 0$ for $1 \leq i \leq s$ and a complete linear system $|NC_1|$ with $N \gg 0$ defines a $\mathbb{P}^1$-fibration $\rho : V \to B$. It is clear that the G-action on V preserves the $\mathbb{P}^1$-fibration ρ.

(3) All the exceptional components of $f : V \to \overline{V}$ are the fiber components of the $\mathbb{P}^1$-fibration. Thence follows the assertion. Q.E.D.

Theorem 1.7 *With the above notations and assumptions, suppose that* $(\overline{V}, \overline{\Delta})$ *is G-relatively minimal. Let* $\overline{F}$ *be a G-extremal curve such that* $(\overline{F}^2) = 0$*and let* $\overline{\rho} : \overline{V} \to B$ *be the* $\mathbb{P}^1$*-fibration defined by a linear system* $|n\overline{F}|$ *for* $n \gg 0$ (cf. Lemmas 1.3 and 1.6). *Let* $\overline{R}$ *be a fiber* $\overline{\rho}$*. Then the following assertions hold.*

(1) *A multiple of* $\overline{R}$ *is written in the form* $\sum_{g\in G} \overline{L}^g$ *up to the algebraic equivalence, where* $\overline{L}$ *is an irreducible curve.*

(2) *If* $\overline{R}$ *is reducible and there are no singular points lying on* $\overline{R}$*, then* $\overline{R}$ *consists of two* (-1) *curves.*

(3) *If* $\overline{R}$ *is irreducible and reduced, then* $\overline{R}$ *is a smooth fiber. If* $\overline{R}$ *is non-reduced, there is a singular point lying on* $\overline{R}$*.*

Proof. (1) Since $\overline{R}$ is numerically equivalent to a $\mathbb{Q}$-multiple of $\overline{F}$, a multiple of $\overline{R}$ is a divisor of the form $\sum_{g\in G} \overline{L}$, where $\overline{L}$ is an irreducible curve.

(2) Write $\overline{R} = \sum_{i=1}^{s} \overline{L}_i$. If $s \geq 2$, then $(\overline{L}_i^2) < 0$ for every i because $\overline{R}$ is connected. The assertion (1) implies that $(\overline{L}_i \cdot \overline{\Delta} + K_{\overline{V}}) < 0$ because $(\overline{F} \cdot \overline{\Delta} + K_{\overline{V}}) < 0$ and the irreducible components of $\overline{R}$ are G-translates. If there are no singular points lying on $\overline{R}$, the arguments in Lemma 1.6 shows that $\overline{R}$ consists of two (-1) curves.

(3) If $\overline{R}$ is irreducible and if there are singular points lying on $\overline{R}$, the proper trnsform $R := f'(\overline{R})$ is a unique (-1) curve in a fiber $f^*(\overline{R})$ of the $\mathbb{P}^1$-fibration $\overline{\rho} \cdot f : V \to C$. So, the multiplicity of R is greater than 1. The assertion (3) follows from this observation. Q.E.D.

We shall next consider the case $\rho(\overline{V}//G) = 1$. Then $\overline{\mathrm{NE}}\,(\overline{V})^G = \mathbb{R}_+[\overline{F}]$ with $(\overline{F}^2) > 0$. Furthermore, $\overline{W} = \overline{V}//G$ is a log del Pezzo surface of rank one. If $\overline{\Delta} \neq 0$, then $(\overline{W}, \overline{\Gamma})$ is an open log del Pezzo surface of rank one. Hence $\overline{W} - \overline{\Gamma}$ is either affine-ruled or isomorphic to $\mathbb{A}^2/\widetilde{G}$, where $\widetilde{G}$ is a small finite subgroup of GL $(2, k)$. If $\overline{\Delta} = 0$, then $\overline{V}$ is a complete log del Pezzo surface of rank one. These cases will be treated elasewhere in details.

Example **1.8** *Let $\widetilde{G}$ be a small finite subgroup of* $\mathrm{GL}(2, k)$*, which is a central extension*

$$0 \longrightarrow C_a \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow (1),$$

where C_a is a cyclic group of order a and G is a finite subgroup of PGL $(2, k)$*. The natural action of $\widetilde{G}$ on $\mathbb{A}^2$ via* $\mathrm{GL}(2, k)$ *extends to an action of $\widetilde{G}$ onto $\mathbb{P}^2$, where $\mathbb{A}^2$ is embedded into $\mathbb{P}^2$ via $(x, y) \mapsto (x, y, 1)$. Let H_0 be the hyperplane at infinity. Then H_0 is $\widetilde{G}$-stable. Set $\overline{V} = \mathbb{P}^2/C_a$ and let $\overline{\Delta}$ be the image of H_0 on $\overline{V}$. Then $(\overline{V}, \overline{\Delta})$ is an open log del Pezzo surface of rank one with G-action. The quotient $\overline{W} = \overline{V}//G$ is a completion of $\mathbb{A}^2/\widetilde{G}$.*

Example **1.9** *Let $\overline{V}$ be the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$. Let ι be the involution on $\overline{V}$ which exchanges the $\mathbb{P}^1$-factors, $(P, Q) \mapsto (Q, P)$ for $P, Q \in \mathbb{P}^1$. Then the quotient $\overline{W} = \overline{V}//\langle\iota\rangle$ is isomorphic to $\mathbb{P}^2$, and the quotient morphism $\pi : \overline{V} \to \overline{W}$ branches over a conic $\overline{\Gamma}$ on $\mathbb{P}^2$. Let $\overline{\Delta}$ be the diagonal on $\overline{V}$ such that $\pi^*(\overline{\Gamma}) = 2\overline{\Delta}$. Then $(\overline{W}, \overline{\Gamma})$ is a log del Pezzo surface of rank one, while* rank Pic $(\overline{V}) = 2$.

2 $\mathbb{P}^1$-fibrations preserved by G

Let us begin with the following example which will validate Theorem 2.2. below.

Example **2.1** *Let W_0 be the Hirzebruch surface of degree n and let M_0 (resp. M_1) be the minimal cross-section (resp. a cross-section disjoint from M_0). Let P_0 be a point on M_1 and let ℓ_0 be the fiber passing through P_0. Blow up the point P_0 and its infinitely near points $P_1, \ldots, P_{m-1}$ lying on the fiber ℓ_0. Let $\sigma : W \to W_0$ be the composite of these blowing-ups. Let E_i $(1 \leq i \leq m)$ be the proper transform of the exceptional curve arising from the blowing-up of the point P_{i-1}. Let L_0 be the proper transform of ℓ_0. Then $L_0 + E_1 + 2E_2 + \cdots + (m-1)E_{m-1} + mE_m$ is the total transform of the fiber*

ℓ_0, whose dual graph is a linear chain with $({E_i}^2) = -2$ for $1 \le i \le m-1$, $({E_m}^2) = -1$ and $({L_0}^2) = -m$. Then we can contract the curve L_0 and the curves $E_1 + \cdots + E_{m-1}$ to the cyclic quotient singular points Q_0 and Q_1, respectively. Let $\tau : W \to \overline{V}$ be the contraction. Let $\overline{M}_0 = \tau(M_0)$ and $\overline{M}_1 = \tau(M_1)$. Then $\overline{V}$ has a $\mathbb{P}^1$-fibration $\overline{\rho} : \overline{V} \to \overline{B}$, where $\overline{B} \cong \mathbb{P}^1$. The curves $\overline{M}_0$ and $\overline{M}_1$ are the cross-sections of $\overline{\rho}$, and the point Q_i lies on $\overline{M}_i$ for $i = 0, 1$.

Choose another fiber ℓ_1 of the surface W_0. Then we have a linear equivalence

$$L_0 + E_1 + 2E_2 + \cdots + (m-1)E_{m-1} + (m-1)\sigma^*(\ell_1) \sim m(\sigma^*(\ell_1) - E_m).$$

So, we can consider a degree m cyclic covering $\alpha : \widetilde{W} \to W$ which ramifies over $L + E_1 + \cdots + E_{m-1} + \sigma^*(\ell_1)$. Let $\mu : \widetilde{V} \to \widetilde{W}$ be the minimal resolution of the singularities. Let $\widetilde{L}_0$ and $\widetilde{E}_m$ be respectively the reduced inverse images of L_0 and E_m. Then $(\widetilde{L}_0^2) = -1$ and $(\widetilde{E}_m^2) = -m$. In fact, we can show that $(\alpha \cdot \mu)^{-1}(E_1 + \cdots + E_{m-1})$ and $\widetilde{L}_0$ are contractable to smooth points. Let $\nu : \widetilde{V} \to V$ be the contraction. Then V has a $\mathbb{P}^1$-fibration $\rho : V \to B$ such that R_i, which is the proper transform on V of M_i on W_0, is a cross-section of ρ for $i = 1, 2$ and B is an m-ple covering of $\overline{B}$ totally ramifying over the points $\overline{\rho}(\ell_i)$ $(i = 0, 1)$. The cyclic group G of order m acts on $\widetilde{V}$ as the covering transformation group and the action of G descends down to V. By the above construction, we know that the quotient surface $V//G$ is isomorphic to $\overline{V}$. Note that if $g \neq e$ the g-translate of a general fiber ℓ of ρ is a fiber different from ℓ.

We can consider the degenerate fibers of the same type $L_0^{(i)} + E_1^{(i)} + 2E_2^{(i)} + \cdots + (m-1)E_{m-1}^i + mE_m^i$ for $i = 1, \ldots, r$ and the smooth fibers $\sigma^*(\ell_j)$ for $j = r+1, \ldots, r+s$, where $r + s$ is an even integer. Then the curve B above is a smooth projective curve of genus $(m-1)(r+s-2)/2$.

A $\mathbb{P}^1$-fibration $\rho : V \to B$ from a smooth projective surface V with a finite group G-action is called *G-relatively minimal* if G preserves the fibration ρ and if every fiber is irreducible unless it consists of two (-1) curves $C + C'$ with $C' = C^g$ for some $g \in G$. We assume further that V with a reduced effective divisor D is a minimal resolution of $(\overline{V}, \overline{\Delta})$ as in section one. The next result shows that this example is essentially the unique case of a G-relatively minimal $\mathbb{P}^1$-fibration with a horizontal irreducible component of D and a finite group G acting only along fibers.

Theorem 2.2 *Let $\rho : V \to B$ be a $\mathbb{P}^1$-fibration defined by an extremal curve F as in Lemma 1.6. Suppose that (V, ρ) is relatively minimal in the above sense and that D contains horizontal components. Then the following assertions hold.*

(1) *The $\mathbb{P}^1$-fibration $\rho : V \to B$ is a $\mathbb{P}^1$-bundle. Namely, there are no fibers consisting of two (-1) curves.*

(2) *There is only one horizontal component, say R, of D which is not a component of* $\mathrm{Bk}\, D$*, G-stable and a cross-section of ρ.*

(3) *Let $\alpha : G \to \mathrm{Aut}\, B$ be the natural group homomorphism. If α is injective, the quotient surface $\overline{V} := V//G$ is a normal projective surface with a $\mathbb{P}^1$-fibration over $\overline{B} := B//G$ and V is isomorphic to the normalization of the fiber product $\overline{V} \times_{\overline{B}} (B, q)$, where $q : B \to \overline{B}$ is the quotient morphism.*

(4) *Let H be the kernel of the homomorphism α. Then H is a cyclic group. Let $V_1 := V//H$ be the quotient surface. Then V_1 is a smooth projective surface with a $\mathbb{P}^1$-fibration $\rho_1 : V_1 \to B$. The quotient morphism $p_1 : V \to V_1$ is a cyclic covering which totally ramifies over the two cross-sections of the $\mathbb{P}^1$-fibration ρ_1, and D is one of these two disjoint sections.*

Proof. (1) Suppose that $F = C + C'$ is a fiber of ρ consisting of two (-1) curves. Let R be a horizontal component of D. Suppose $R \not\subset \mathrm{Supp}\, \mathrm{Bk}\, D$. Then $R^g \not\subset \mathrm{Supp}\, \mathrm{Bk}\, D$ for any $g \in G$. Note that $C' = C^g$ for some $g \in G$ and $(F \cdot R) > 0$. If R is G-stable, then $(F \cdot D^{\#}) \geq (R \cdot F) = 2(R \cdot C) \geq 2$. If $R^g \neq R$ for some $g \in G$, then $(F \cdot D^{\#}) \geq (F \cdot R + R^g) \geq 2$. So we always have $(F \cdot D^{\#} + K_V) = -2 + (F \cdot D^{\#}) \geq 0$. This is a contradiction because $\sum_{g \in G} F^g$ generates an extremal ray of $\overline{\mathrm{NE}}\,(\overline{V})^G$. Hence any horizontal component R of D is a component of $\mathrm{Bk}\, D$. Let $\{R_i\}_{i \in I}$ be the set of all irreducible components of $\mathrm{Bk}\, D$. Define the rational numbers α_i by the condition:

$$(F + \sum_{i \in I} \alpha_i R_i \cdot R_i) = 0 \quad \text{for every} \quad i \in I.$$

Then $\sum_{i \in I} \alpha_i R_i$ is a nonzero effective divisor because the intersection matrix of $\mathrm{Bk}\, D$ is negative definite and $\mathrm{Bk}\, D$ contains at least one horizontal component. Furthermore, the coefficient $\alpha_i > 0$ if $(R_i \cdot F) > 0$. Then we have

$$(\overline{F}^2) = ((F + \sum_{i \in I} \alpha_i R_i)^2) = (F \cdot F + \sum_{i \in I} \alpha_i R_i) > 0,$$

which contradicts the hypothesis that $(\overline{F}^2) = (F^2) = 0$. Hence ρ is a $\mathbb{P}^1$-bundle.

(2) By (1) above, any fiber F of ρ is irreducible. If there is an irreducible horizontal component of Bk D, then $(\overline{F}^2) > 0$ as shown above, which is a contradiction. Suppose either that there exists an irreducible component R of D with $(R \cdot F) \geq 2$ or that R is not G-stable. Then we have a contradiction as in (1).

(3) Let $Y = V - R$, where R is the unique G-stable cross-section of ρ. By (2) above, Y is G-stable and $\rho\,|_Y \colon Y \to B$ is an $\mathbb{A}^1$-bundle. We denote $\rho\,|_Y$ by the same letter ρ. Since G preserves the $\mathbb{A}^1$-fibration $\rho : Y \to B$, there exists a natural group homomorphism $\alpha : G \to \mathrm{Aut}\, B$ such that $\rho(v)^{\alpha(g)} = \rho(v^g)$ for $g \in G$ and $v \in V$. Let K be the function field of B over the ground field k and let $\overline{K}$ be the function field of $\overline{B}$ over k, where $\overline{B} = B//G$. Since the generic fiber of ρ is the affine line $\mathbb{A}^1_K := \mathrm{Spec}\, K[x]$, we have

$$g(x) = a(g)x + b(g) \quad \text{with} \quad a(g) \in K^*, b(g) \in K \quad \text{for} \quad g \in G,$$

where

$$a(gh) = a(h)^g a(g) \quad \text{and} \quad b(gh) = a(h)^g b(g) + b(h)^g \quad \text{for} \quad g, h \in G.$$

Suppose that α is injective. Then G is considered as the Galois group of a field extension $K/\overline{K}$. By Theorem 90 of Hilbert, there exists $c \in K^*$ such that $a(g) = c \cdot (c^g)^{-1}$. Replacing x by cx, we may assume that $a(g) = 1$. Then $b(g) = (d - d^g)/|G|$, where $d = \sum_{g \in G} b(g)$. Replacing x by $x + d/|G|$, we may assume that $b(g) = 0$ as well. Hence, with x chosen this way, we have $g(x) = x$ for $g \in G$.

It then follows that there exists a G-stable open set U of B such that $\rho^{-1}(U) \cong U \times \mathbb{P}^1$ with G acting on the factor U. Set $\overline{V} := V//G$, which is a normal projective surface with the morphism $\overline{\rho} : \overline{V} \to \overline{B}$, where $\overline{B} := B//G$. Let $p : V \to \overline{V}$ and $q : B \to \overline{B}$ be the quotient morphisms. Then $\overline{\rho} \cdot p = q \cdot \rho$. The above observation implies that $\overline{\rho}^{-1}(\overline{U}) \cong \overline{U} \times \mathbb{P}^1$, where $\overline{U} = U//G$. Hence $\overline{\rho} : \overline{V} \to \overline{B}$ is a $\mathbb{P}^1$-fibration. Let $\overline{R} := R//G$. Since R is a cross-section of ρ, it follows that $\overline{R}$ is a cross-section of $\overline{\rho}$, i.e., $\overline{\rho}\,|_{\overline{R}}$ is the identity morphism. By [6], the singularities of $\overline{V}$ which are not on the cross-section $\overline{R}$ are cyclic quotient singularities, for $\overline{V} - \overline{R}$ contains an $\mathbb{A}^1$-cylinder. It is clear that the natural morphism $V \to \overline{V} \times_{\overline{B}} B$ is a finite birational morphism. Hence V is isomorphic to the normalization of the fiber product $\overline{V} \times_{\overline{B}} B$.

Note that if a fiber $\overline{F}$ has a singular point Q of $\overline{V}$ which is not on the cross-section $\overline{R}$, then $\overline{V}$ has also a singular point on the point $\overline{F} \cap \overline{R}$. In fact, let Q be a singular point of $\overline{V}$ not lying on $\overline{R}$ if it exists at all. Suppose that $\overline{F} \cap \overline{R}$ is a smooth point of $\overline{V}$. Since $\rho : V \to B$ is a $\mathbb{P}^1$-bundle, $\overline{F}$ is irreducible. Let $\sigma : W \to \overline{V}$ be the minimal resolution of singularities. Then $\sigma^{-1}(Q)$ consists of rational curves, each component of which has self-intersection number ≤ -2. Hence $\sigma^{-1}(\overline{F})$ consists of the proper transform F' of $\overline{F}$ and the rational curves with self-intersection number ≤ -2. Since $\sigma^{-1}(\overline{F})$ is a degenerate fiber of a $\mathbb{P}^1$-fibration $\overline{\rho} \cdot \sigma : W \to \overline{B}$, the component F' is a unique (-1) component. Since $\overline{F}$ meets the cross-section $\overline{R}$ and since $\overline{F} \cap \overline{R}$ is a smooth point of $\overline{V}$, F' has multiplicity 1 in the fiber $\sigma^{-1}(\overline{F})$. Then there exists another (-1) component in $\sigma^{-1}(\overline{F})$. This is a contradiction.

(4) With the notations in the proof of (3), the correspondence $g \mapsto a(g)$ induces a group homomorphism $\beta : H \to G_m$, where $H = \operatorname{Ker} \alpha$. If the homomorphism α is not injective, then we have, with the above notations, $g(x) = x + b(g)$ for $g \in \operatorname{Ker} \beta$. Then g is of infinite order provided $g \in \operatorname{Ker} \beta$ and $g \neq 1$, for $b(g) \neq 0$. This is a contradiction. Hence $\operatorname{Ker} \beta = (1)$. So, β is injective. Then, as a finite subgroup of G_m, H is a cyclic group of order, say n. Let g be a generator of the group H. Then $a(g)$ is an n-th primitive root of unity. Suppose $H \neq (1)$. Then $g(x) = a(g)x + b(g)$ and $g(x+c) = a(g)(x+c)$, where $c = b(g)/(a(g) - 1)$. Then the point $x = -c$ is left fixed under the H-action. The stated assertion then follows immediately. Q.E.D.

We consider next the case where the boundary divisor D has no horizontal components with respect to a $\mathbb{P}^1$-fibration $\rho : V \to B$. Our objective is to prove Theorem 2.6.

Lemma 2.3 *Let $\rho : V \to C$ be a $\mathbb{P}^1$-fibration which is preserved by G. Suppose that ρ contains a fiber of type $F = C + C'$, where $C' = C^g$ for some $g \in G$. Then the following assertions hold:*

(1) *$C = C'^g$ and $C \cap C'$ is fixed under the action of g.*

(2) *If g acts along the fibers, i.e., $\rho \cdot g = \rho$, then g has order* 2 *or* 4.

(3) *If g^2 moves the fibers, i.e., $\rho\cdot g \neq \rho$, then g^2 has order either $n = 2(s+1)$ for some $s > 0$ or $n = 3$.*

Proof. (1) Suppose that C'^g is contained in a fiber different from F. Since C^g is contained in the same fiber as C'^g, it follows that C'^g is contained in the fiber F. Hence $C'^g = C$. Let $P := C \cap C'$. Then $P^g \in C^g \cap C'^g = C' \cap C = \{P\}$, whence we know that P is fixed under the action of g. Note that this does not necessarily imply that g has order 2. In fact, if one writes $g_*(\xi) = \lambda\xi'$ and $g_*(\xi') = \mu\xi$ with $\lambda, \mu \in k^*$, where ξ and ξ' are respectively the tangential directions of C and C' at the point P, it may occur that $\lambda \neq \mu$.

(2) The proof of the assertion consists of three steps.

(i) By the assertion (1), we have $C^{g^2} = C$ and $C'^{g^2} = C'$. Set $h = g^2$. Suppose that h is not the identity element. Then h is an automorphism of $\mathbb{A}^1 = C - \{P\}$. Let z be an inhomogeneous coordinate of $\mathbb{A}^1$, with respect to which the point P corresponds to $z = \infty$. Then $h(z) = \alpha z + \beta$ with $\alpha \in k^*$ and $\beta \in k$. Let n be the order of h. It is then easy to show that

$$\alpha^n = 1 \quad \text{and} \quad (\alpha^{n-1} + \cdots + \alpha + 1)\beta = 0.$$

Hence $\alpha = 1$ implies $\beta = 0$. So, $\alpha \neq 1$ if $h^2 \neq 1$. Let $\gamma = \beta/(\alpha - 1)$. Then $h(z + \gamma) = \alpha(z + \gamma)$. So, by replacing z by $z + \gamma$, we may assume that $h(z) = \alpha z$. This implies that the components C and C' have the points Q and Q' respectively which are fixed under h.

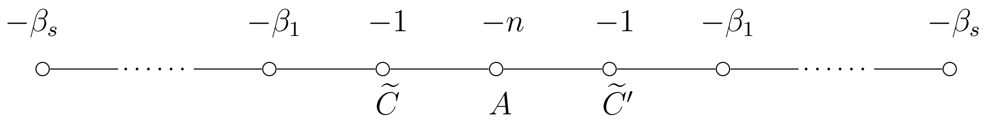
(ii) In an open neighborhood of the point P, we can choose a system of local coordinates (x, y) such that C (resp. C') is defined by $y = 0$ (resp. $x = 0$) and that $h(x) = \alpha x$ and $h(y) = \alpha y$. Here h acts on x and y with the same weight because C and C' are interchanged by the action of g. Let $q : V \to \widehat{V}$ be the quotient morphism under the group $\langle h\rangle$. Then $\widehat{V}$ has a $\mathbb{P}^1$-fibration $\widehat{\rho} : \widehat{V} \to \widehat{B}$ because h preserves the $\mathbb{P}^1$-fibration ρ. Set $\widehat{P} = q(P), \widehat{Q} = q(Q)$ and $\widehat{Q}' = q(Q')$. Then the branch locus of q contains these three points $\widehat{P}, \widehat{Q}$ and $\widehat{Q}'$. Meanwhile, $\widehat{V}$ has an isolated cyclic quotient singularity at $\widehat{P}$. Hence $\widehat{P}$ is the isolated component of the branch locus of q. The minimal resolution of $\widehat{P}$ consists of only one $(-n)$ curve A and the proper transforms of $q(C), q(C')$ (say, $\widehat{C}, \widehat{C}'$, respectively) meet the component A transversally.

(iii) On the other hand, the points $\widehat{Q}$ and $\widehat{Q}'$ are smooth points, for the cyclic group $\langle h\rangle$ acts on V near Q as $h(\xi,\eta)=(\alpha^{-1}\xi,\eta)$ with respect to a suitable local system of parameters (ξ,η). Hence $\widehat{C}+A+\widehat{C}'$ is a degenerate fiber of the $\mathbb{P}^1$-fibration on a smooth surface. This implies that $\widehat{C}$ and $\widehat{C}'$ are (-1) curves. Hence $n=2$. Hence the order of g is 4.

(3) Suppose that h moves the fibers. Then the points Q and Q' are the isolated fixed points of h. Suppose that h acts on V near Q as $h(\xi,\eta)=(\alpha^{-1}\xi,\alpha^{-d}\eta)$ with $0<d<n$, where (ξ,η) is a suitable local system of parameters at Q. Let $e=\gcd(n,d)$ and let

$$\frac{n}{d}=\beta_1-\cfrac{1}{\beta_2-\cfrac{1}{\ddots\ -\cfrac{1}{\beta_s}}}$$

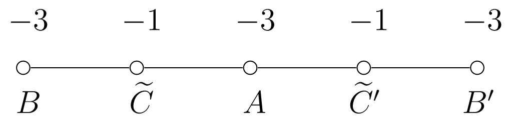
be the continued fraction expansion of $(n/e)/(d/e)$. Then the minimal resolution of three singular points $\widehat{P},\widehat{Q},\widehat{Q}'$ together with the proper transforms $\widetilde{C},\widetilde{C}'$ of $\widehat{C},\widehat{C}'$ give rise to a degenerate $\mathbb{P}^1$-fiber whose dual graph is given as follows.



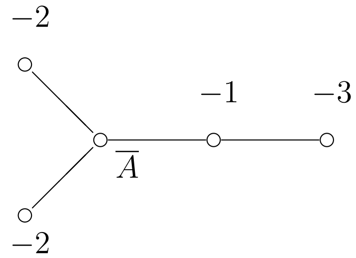
Hence either $\beta_1=\cdots=\beta_s=2$ and $n=2(s+1)$ or $n=\beta_1=3$ and $s=1$. In the first case, $n/e=s+1$ and $d/e=s$. Hence $e=2,n=2(s+1)$ and $d=2s$. In the second case, $n=3$ and $d=1$. Q.E.D.

Lemma 2.4 *In Lemma 2.3, the case where* g^2 *has order* 3 *is impossible.*

Proof. The element g induces an involution $\widehat{g}$ on $\widehat{V}$ which preserves the $\mathbb{P}^1$-fibration $\widehat{\rho}$ and interchanges $\widehat{C}$ and $\widehat{C}'$. Hence $\widehat{g}$ fixes the point $\widehat{P}$ and interchanges the points $\widehat{Q}$ and $\widehat{Q}'$. Hence $\widehat{g}$ lifts to an involution $\widetilde{g}$ on the minimal resolution $\sigma:\widetilde{V}\to\widehat{V}$. The $\mathbb{P}^1$-fibration $\widehat{\rho}$ lifts to a $\mathbb{P}^1$-fibration $\widetilde{\rho}:\widetilde{V}\to\widehat{B}$. The inverse image of the fiber $\widehat{C}+\widehat{C}'$ has the following linear chain:



where we denote the proper transforms of $\widehat{C}$ and $\widehat{C}'$ on $\widetilde{V}$ by $\widetilde{C}$ and $\widetilde{C}'$, respectively. The exceptional curve A arising from $\widehat{P}$ is stable under the action of $\widetilde{g}$. Since $\widetilde{g}$ interchanges the points $\widetilde{C} \cap A$ and $\widetilde{C}' \cap A$, there are two points R, R' on A which are fixed by $\widetilde{g}$. We consider two cases according as $\widetilde{g}$ moves the fibers of $\widetilde{\rho}$ or not. Let $\overline{W}$ be the quotient of $\widetilde{V}$ by $\widetilde{g}$ which has the $\mathbb{P}^1$-fibration $\overline{\rho}' : \overline{W} \to \overline{B}'$, where $\overline{B}' = \widehat{B}//\langle \widetilde{g} \rangle$. Let $\overline{F}$ be the image of $\sigma^{-1}(\widehat{C} + \widehat{C}')$. Suppose $\widetilde{g}$ moves the fibers of $\widetilde{\rho}$. Then the points R, R' are mapped to the singular points $\overline{R}, \overline{R}'$ of type A_2 on the surface $\overline{W}$. Then the minimal resolution of the points $\overline{R}, \overline{R}'$ gives rise to a degenerate $\mathbb{P}^1$-fiber of the following type on a smooth surface:



But, with whatever value for the self-intersection number $(\overline{A}^2)$, this graph cannot be the graph of a degenerate $\mathbb{P}^1$-fiber. Suppose that $\widetilde{g}$ does not move the fibers of $\widetilde{\rho}$. Then there are no singular points appearing on the fiber $\overline{F}$. Since $(A^2) = -3$ and $(A^2) = 2(\overline{A}^2)$, this is impossible again. Q.E.D.

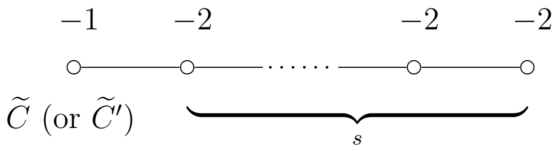
Lemma 2.5 *In Lemma 2.3, the following two cases do not occur.*

(1) *The element g^2 has order $n = 2(s+1)$ with $s > 0$.*

(2) *The element g has order* 4.

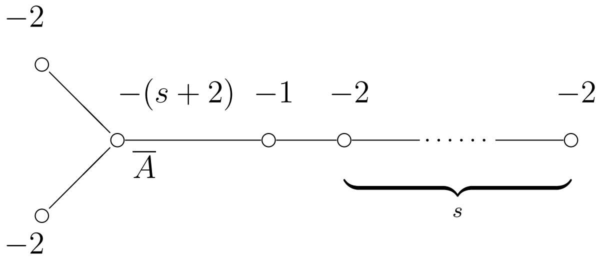
Proof. (1) With the notations in Lemmas 2.3 and 2.4, the points Q, Q' on C, C' are isolated fixed points under the action of $h := g^2$. Hence the points $\widehat{Q}, \widehat{Q}'$ on $\widehat{C}, \widehat{C}'$ are the isolated cyclic singular points on the surface $\widehat{V}$. The element g induces an automorphism on $\widehat{V}$ which we denote by $\widehat{g}$. Then $\widehat{g}$

lifts to an automorphism $\widetilde{g}$ on the minimal resolution $\widetilde{V}$ which interchanges the components $\widetilde{C}, \widetilde{C}'$ as well as the linear chains sprouting from $\widetilde{C}, \widetilde{C}'$.



Furthermore, $\widetilde{g}$ acts on the component A. Since $\widetilde{g}$ interchanges the points $\widetilde{C} \cap A, \widetilde{C}' \cap A$ on A, there are two other points R, R' of A which are left fixed by $\widetilde{g}$. There are two cases according as $\widetilde{g}$ moves the fibers of the $\mathbb{P}^1$-fibration $\widetilde{\rho} : \widetilde{V} \to \widehat{B}$ or not.

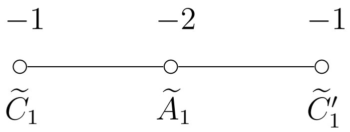
Case 1. Consider first the case where $\widetilde{g}$ moves the fibration $\widetilde{\rho}$. Let $\overline{W}$ be the quotient of $\widetilde{V}$ by $\widetilde{g}$. Then the images $\overline{R}, \overline{R}'$ of R, R' are the cyclic singular points of type A_2. Let W be the minimal resolution of $\overline{W}$. Then W has a $\mathbb{P}^1$-fibration $\rho_W : W \to \overline{B}'$, where $\overline{B}'$ is the quotient of $\widehat{B}$ by the involution induced by $\widetilde{g}$. The fiber of ρ_W corresponding to $\widehat{C} + \widehat{C}'$ on $\widehat{V}$ has the following dual graph:



In fact, the surface $\widetilde{V}$ near the concerned fiber is obtained by taking a double covering of W ramifying on the two (-2) components meeting $\overline{A}$ and one more fiber of ρ_W and by contracting the inverse images of the (-2) components which become (-1) components on the double covering.

Since the $\mathbb{P}^1$-fibation ρ_W is trivial over an open set of $\overline{B}'$, the function field $k(\widetilde{V})$ is written as $k(W)[t]/(t^2 = x)$, where x is an inhomogeneous parameter of $\overline{B}'$ such that the above double covering ramifies over the fibers (the fiber components) of ρ_W lying over $x = 0, \infty$.

To go further, we have to look into the quotient morphism $q : V \to \widehat{V}$ more closely. With the notations of Lemma 2.3, the element h acts on V near the point Q as $h(\xi, \eta) = (\alpha^{-1}\xi, \alpha^{-d}\eta)$, where α is a primitive n-th root of unity and $d = 2s$. Set $h' = h^{s+1}$. Then ${h'}^2 = 1$ and the element h' acts as $h'(\xi, \eta) = (-\xi, \eta)$ near the point Q (and hence near the point Q'). Hence h' acts along the fibers of ρ, i.e., $\rho \cdot h' = \rho$. Let $q_1 : V \to \widehat{V}_1$ be the quotient morphism by h' and let $\widetilde{V}_1$ be the minimal resolution of $\widehat{V}_1$. The $\mathbb{P}^1$-fibration $\rho : V \to B$ descends to a $\mathbb{P}^1$-fibration $\widetilde{\rho}_1 : \widetilde{V}_1 \to B$ and the fiber $\widetilde{\rho}_1^{-1}(\rho(C + C'))$ has the following dual graph:



Let $\widehat{h}_1$ and $\widetilde{h}_1$ be respectively the automorphism on $\widehat{V}_1$ and $\widetilde{V}_1$ induced by h. Then $\widehat{h}_1^{s+1} = 1$ (resp. $\widetilde{h}_1^{s+1} = 1$) and $\widetilde{h}_1$ acts trivially on the component $\widetilde{A}_1$. Let Q_1, Q'_1 be the images on $\widetilde{V}_1$ of the points Q, Q', respectively. Then $\widetilde{h}_1$ acts on $\widetilde{V}_1$ near the point Q_1 (and Q'_1) as $\widetilde{h}_1(\xi_1, \eta_1) = (\alpha_1^{-1}\xi_1, \alpha_1^{-s}\eta_1)$, where α_1 is a primitive $(s+1)$-th root of unity and where (ξ_1, η_1) is a system of local parameters such that $\widetilde{C}_1$ is defined by $\eta_1 = 0$. This implies that the quotient morphism $B \to \widehat{B} := B/\langle h \rangle$ has degree $s + 1$. The quotient of $\widehat{V}_1$ by the action of $\widehat{h}_1$ is the surface $\widehat{V}$. Hence the function field $k(\widehat{V}_1)$ is given as $k(\widehat{V}) \otimes_{k(\widehat{B})} k(B)$ with a cyclic Galois extension $k(B)/k(\widehat{B})$ of degree $s + 1$ and the function field $k(V)$ is obtained as $k(\widehat{V}_1)[u]/(u^2 = y)$, where y is a fiber coordinate of the $\mathbb{P}^1$-fibration $\widetilde{\rho}_1$ over an open set of B. Thus the field extension $k(V)/k(\overline{W})$ is a composite of a cyclic Galois extension $k(\widehat{V}_1) = k(\overline{W}) \otimes_{k(\overline{B}')} k(B)$ of degree $2(s+1)$ and a quadratic extension $k(\widehat{V}_1)[u]/(u^2 = y)$. Since $k(V) = k(B)(u)$ and $k(u)$ is linearly disjoint from $k(B)$ over k, the field extension $k(V)/k(\overline{W})$ cannot be a cyclic extension of degree $4(s+1)$.

Case 2. Consider next the case where $\widetilde{g}$ does not move the fibration $\widetilde{\rho}$. With the notations of Case 1 above, the $\mathbb{P}^1$-fibration $\widetilde{\rho} : \widetilde{V} \to \widehat{B}$ has two horizontal cross-sections which meet the fiber components A in the points R, R'. Hence the function field $k(\widehat{V})$ is a quadratic extension $k(\overline{W})[u]/(u^2 = z)$, where z is a fiber coordinate of the $\mathbb{P}^1$-fibration $\rho_W : W \to \overline{B}'$. Here we have $\overline{B}' = \widehat{B}$ because $\widetilde{g}$ induces the identity on $\widehat{B}$. Let $q_2 : \widehat{V}_1 \to \widehat{V}$ be the quotient morphism by the automorphism $\widehat{h}$ induced by h. The same arguments as in the case 1 shows that the function field $k(\widehat{V}_1)$ is written as $k(\widehat{V}) \otimes_{k(\widehat{B})} k(B)$, where $k(B)/k(\widehat{B})$ is a cyclic extension of degree $s+1$. Furthermore, q_2 induces an isomorphism between the comonents $\widetilde{A}_1$ and A lying on the minimal resolutions $\widetilde{V}_1$ and $\widetilde{V}$ of $\widehat{V}_1$ and $\widehat{V}$, respectively. Let R_1, R'_1 be the points of $\widehat{A}_1$ which correspond to the points R, R'. We may assume that $u = 0, \infty$ at the points R, R', respectively. Hence, if we consider u as the inhomogeneous coordinate of $\widetilde{A}_1$ via q_2, u takes the value $0, \infty$ at the points R_1, R'_1, respectively. Note that the points R_1, R'_1 are different from the points $\widetilde{C}_1 \cap \widetilde{A}_1, \widetilde{C}'_1 \cap \widetilde{A}_1$. So, we may assume that $u = 1$ (resp. -1) at the point $\widetilde{C}_1 \cap \widetilde{A}_1$ (resp. $\widetilde{C}'_1 \cap \widetilde{A}_1$). The double covering $q_1 : V \to \widehat{V}_1$ ramifies over $\widetilde{A}_1$ and the two cross-sections of $\widetilde{\rho}_1$ which meet the fiber $\widetilde{C}_1 + \widetilde{A}_1 + \widetilde{C}'_1$ at the points Q_1, Q'_1 on the components $\widetilde{C}_1, \widetilde{C}'_1$, respectively. More precisely, q_1 extends to a double covering $\widetilde{q}_1 : U \to \widetilde{V}_1$, where U is the blowing-up of V at $C_1 \cap C'_1$, and $\widetilde{q}_1$ ramifies over $\widetilde{A}_1$. Hence the field extension $k(V)/k(\widehat{V}_1)$ is given as

$$k(V) = k(\widehat{V}_1)[v], \quad \text{where} \quad v^2 = \frac{u+1}{u-1}.$$

Hence, replacing v by $(u-1)v$, we may assume that $v^2 = u^2 - 1 = z - 1$. So, we have

$$k(V) = \left(k(\overline{W}) \otimes_{k(\widehat{B})} k(B)\right) [u, v] \,/\, \left(u^2 = z, v^2 = z - 1\right).$$

This implies that the Galois group of the extension $k(V)/k(\overline{W})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{s+1}$, which is not a cyclic group of order $4(s+1)$.

(2) By the proof of Lemma 2.3, this is the case 2 above, where the quotient morphism $q_2 : \widehat{V}_1 \to \widehat{V}$ is the identity morphism. Hence $k(V) = k(\overline{W})[u,v]/(u^2 = z, v^2 = z-1)$, and the Galois group of the extension $k(V)/k(\overline{W})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. So, it is not a cyclic group of order 4. Q.E.D.

As a consequence of Lemmas 2.3, 2.4 and 2.5, we obtain the following result.

Theorem 2.6 *Let $\rho : V \to B$ be a $\mathbb{P}^1$-fibration which is preserved by G. Suppose that ρ contains a fiber of type $F = C + C'$, where $C' = C^g$ for some $g \in G$. Then g is an involution.*

The following example shows a typical way of constructing a smooth projective surface with an involution acting along fibers.

EXAMPLE **2.7** *Let V_0 be the Hirzebruch surface of degree $n \geq 2$ and let M be the minimal section. Let B be a smooth irreducible curve which is linearly equivalent to $2M + 2n\ell$, where ℓ is a fiber of the canonical $\mathbb{P}^1$-fibration ρ_0 on V_0. Then B has genus $n-1$. Hence the restriction $\rho_0|_B : B \to \mathbb{P}^1$, which is a double covering, ramifies at $2n$ points on B. Let $\sigma : V \to V_0$ be the double covering ramified over the curve B and let $\iota : V \to V$ be the covering involution.*

Let $\ell_1, \ldots, \ell_{2n}$ be the fibers of ρ_0 which meet the curve B only in single points with multiplicity 2. Then $\sigma^*(\ell_i) = C_i + C_i'$ $(1 \leq i \leq 2n)$, where C_i and C_i' are the (-1) curves meeting each other transversally in one point. The composite $\rho := \rho_0 \cdot \sigma : V \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration and has $C_i + C_i'$ as a degenerate fiber. The involution ι exchanges C_i and C_i'.

Since $B \cap M = \emptyset$, the inverse image $\sigma^*(M)$ is a disjoint sum $M_0 + M_1$ with $({M_0}^2) = ({M_1}^2) = -n$. We may assume that the C_i meet M_0 and the C_i' meet M_1. The contraction of $C_1', \ldots, C_{2n}'$ brings the surface V back again to the Hirzebruch surface V_0, The image of M_1 is the section M' of V_0 which is disjoint from the minimal section M. Let $\tau : V \to V_0$ be the contraction of $C_1', \ldots, C_{2n}'$. Then we have

$$\begin{aligned} K_{V_0} &\sim -2M_0 - (n+2)\ell \\ K_V &\sim \tau^*(K_{V_0}) + \sum_{i=1}^{2n} C_i' \\ \tau^*(M') &= M_1 + \sum_{i=1}^{2n} C_i'. \end{aligned}$$

Hence $K_V \sim -M_0 - M_1 - 2\ell$. Let $D = M_0 + M_1$. Then we compute

$$D^{\#} + K_V = \frac{n-2}{n} M_0 + \frac{n-2}{n} M_1 + K_V.$$

Then $C_i + C_i'$ is a ι-invariant extremal curve with $(C_i + C_i' \cdot D^{\#} + K_V) = -4/n$.

3 Degree of the fixed point locus

Our interest lies in the following question.

Question. *Let (X, A) be a smooth polarized projective variety of dimension n. Suppose that $(K_X^n) > 0$. Does there then exist an integer N related to $(K_X^i \cdot A^{n-i})$ with $0 \leq i \leq n$ such that any nontrivial automorphism φ of X has the fixed point locus Γ whose A-degree is less than N ?*

We can answer the question only in the case where X is a curve. The authors are indebted to M. Namba for the following results which might be well-known among the experts.

Theorem 3.1 *Let X be a complete smooth curve of genus g defined over the ground field k of characteristic zero. Let φ be a nontrivial automorphism of finite order d. Then the number of the fixed points of φ is less than or equal to $2g/(d-1)+2$. The equality holds if and only if X and φ are given as follows:*

$$X: \quad y^d = (x-a_1)\cdots(x-a_s) \quad \text{with} \quad s = \frac{2g}{d-1}+2,$$

where $a_1, \ldots, a_s$ are all distinct, and where φ is given by

$$(x, y) \mapsto (x, \zeta y)$$

with a d-th primitive root ζ of the unity.

Proof. Let $\langle\varphi\rangle$ be the cyclic group of order d generated by the given automorphism φ. Let Y be the quotient of X by $\langle\varphi\rangle$ and $\pi : X \to Y$ be the quotient morphism. Let g_0 be the genus of the smooth curve Y. Let $B_\pi = \{q_1, \ldots, q_s\}$ be the branch locus of π. Then $\langle\varphi\rangle$ acts transitively on the set $\pi^{-1}(q_i)$ for $1 \leq i \leq s$. Let $e_1, \ldots, e_s$ be the respective ramification indices over the points $q_1, \ldots, q_s$. By the Riemann-Hurwitz formula, we then have

$$2g-2 = d(2g_0-2) + \sum_{i=1}^{s} \frac{d}{e_i}(e_i - 1).$$

It is written as

$$\sum_{i=1}^{s} \left(1 - \frac{1}{e_i}\right) = \frac{2g-2}{d} + 2 - 2g_0. \tag{1}$$

Let P be a point of X. Then P is fixed by φ if and only if φ is totally ramified over the point $q := \varphi(P)$, i.e., $q \in B_\pi$ and $d = e$ at q. Let n be the number of the fixed points of φ. Then the equality (1) implies that

$$n\left(1 - \frac{1}{d}\right) \leq \frac{2g-2}{d} + 2.$$

So, we obtain the inequality

$$n \leq \frac{2g}{d-1} + 2. \tag{2}$$

Now the above computations show that the equality occurs in the inequality (2) if and only if π ramifies totally over every point of B_π and the curve Y has genus $g_0 = 0$. Hence the curve X is determined as follows:

$$y^d = (x - a_1)\cdots(x - a_n), \quad \text{where} \quad n = \frac{2g}{d-1} + 2. \tag{3}$$

Here $(d-1) \mid 2g$ because n is an integer. Q.E.D.

Now fix the genus g and consider when the maximum value of n is attained.

Corollary 3.2 *Let X be a complete smooth curve of genus g and let φ be a nontrivial automorphism of X of finite order. Then the number of the points left fixed by φ is less than or equal to $2g+2$. The number is equal to $2g+2$ if and only if one of the following cases occur:*

(1) *X is a hyperelliptic curve and φ is the hyperelliptic involution.*

(2) *X is an elliptic curve and φ is the multiplication by (-1).*

(3) *X is a rational curve and $\varphi(x) = \zeta x$, where ζ is a primitive d-th root of the unity and x is an inhomogeneous coordinate.*

Proof. The relation in (3) implies that the maximal value $2g+2$ of n is attained if and only if either $d = 2$ and $g > 0$ or $g = 0$. If $d = 2$ and $g \geq 2$, then X is a hyperelliptic curve and φ is the hyperelliptic involution. Suppose $g = 1$ and $d = 2$. Then the hyperelliptic involution is the multiplication by (-1) when X is defined by the equation in (3). If $g = 0$ then φ fixes two points P_1, P_2. Choose an inhomogeneous coordinate x so that $x = 0, \infty$ at P_1, P_2, respectively. Then $\varphi(x) = \zeta x$ for a primitive d-th root of unity. Q.E.D.

We note that any automorphism φ has finite order provided $g \geq 2$.

Corollary 3.3 *Let X be a complete smooth curve of genus $g \geq 3$. Suppose that X is trigonal. Namely, there exists a degree 3 morphism $f : X \to \mathbb{P}^1$. Let φ be a nontrivial automorphism. Then the number of points left fixed by φ is less than or equal to $g+2$.*

Proof. As remarked above, φ has finite order. If the quotient morphism π coincides with the trigonal morphism f above, then $d = 3$ and $n \leq g + 2$. Otherwise, $d \geq 4$ because X is not hyperelliptic. Q.E.D.

Corollary 3.4 *Let X be a complete smooth curve with genus $g \geq 5$. Let φ be a nontrivial automorphism of X. Suppose that X is not hyperelliptic nor trigonal. Then the number of points left fixed by φ is less than or equal to $(2/3)g + 2$.*

Proof. Clear.

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