EQUIVARIANT DERIVATIONS AND ADDITIVE GROUP ACTIONS

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Abstract. Let G be a reductive algebraic group defined over an algebraically closed field of characteristic 0. Let X be an affine variety with an algebraic G-action. We are interested in the G_a-actions on X commuting with the G-action. We observe G-equivariant (locally nilpotent) derivations in various settings from the viewpoint of the invariant theory.

1. Introduction

Let k be an algebraically closed field of characteristic 0, which is the ground field. The action of the additive group G_a on an affine algebraic variety gives us important information on the underlying variety. Let G be a reductive algebraic group and let $X = \operatorname{Spec} B$ be an affine G-variety. We observe G_a-actions on X commuting with the G-action. It is well-known that G_a-actions on X correspond bijectively to locally nilpotent derivations on B. We say that a derivation δ on B is *G-equivariant* if $\delta \cdot g = g \cdot \delta$ for every $g \in G$. The G_a-action on X induced by a locally nilpotent derivation δ commutes with the G-action if and only if δ is G-equivariant. Let $\pi : X \to X/\!/G = \operatorname{Spec} B^G$ be the algebraic quotient where B^G is the ring of G-invariants. Every G_a-action on X commuting with the G-action induces a G_a-action on $X/\!/G$. However, a G_a-action on $X/\!/G$ does not necessarily lift up to a (G-equivariant or not) G_a-action on X. In this article, we investigate G-equivariant (locally nilpotent) derivations in various settings from the viewpoint of the invariant theory. After reviewing G-actions and locally nilpotent derivations in section 2, we observe G-equivariant (locally nilpotent) derivations on G-representation spaces in sections 3 and 4. In section 4, we focus on the case where G is finite. We see in examples that a (locally nilpotent) derivation on B^G does not necessarily lift up to

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a (locally nilpotent) derivation on B. In section 5, we show that the G-equivariant G_a-action characterizes a smooth acyclic complex affine G-variety X when the dimension of $X//G$ is one or two (Theorems 5.1 and 5.2). In the last section, we consider the case where G is abelian and observe semi-invariant locally nilpotent derivations associated to G-equivariant embeddings. Let V and $\tilde{V}$ be G-representation spaces of dimension m and n, respectively. A G-equivariant embedding $V \hookrightarrow \tilde{V}$ of G-varieties is abbreviated to a G-embedding. We say that a G-embedding $\varphi : V \to \tilde{V}$ is *G-rectifiable* if there exists a system of semi-invariant coordinate functions $f_1, \ldots, f_n$ on $\tilde{V}$ such that the image $\varphi(V)$ is defined by the G-stable ideal $(f_{m+1}, \ldots, f_n)$. Forgetting the G-action, a G-rectifiable embedding $V \to \tilde{V}$ is a rectifiable embedding $\mathbb{A}^m \to \mathbb{A}^n$. It is known in [6] that the embeddings $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$ bijectively correspond to the sequences $\Delta = (\delta_1, \ldots, \delta_m)$ of mutually commuting, locally nilpotent derivations δ_i's of some type on a polynomial ring with a linear action of the algebraic torus G_m. This result extends G-equivariantly (Theorem 6.2, cf. [7]). For a given G-embedding $\varphi : V \to \tilde{V}$, there exsits a sequence $\Delta_\varphi = (\delta_1, \ldots, \delta_m)$ associated to φ where δ_i's are mutually commuting, semi-invariant locally nilpotent derivations of some type on a polynomial ring B with a linear $(G \times G_m)$-action. Let $A = \cap_{i=1}^m B^{\delta_i}$ where B^{δ_i} denotes the kernel of δ_i. Then the G-embedding φ is G-rectifiable if and only if $Y = \operatorname{Spec} A$ is isomorphic to a $(G \times G_m)$-representation space (Theorem 6.3, cf. [7]). If there exists a G-embedding which is rectifiable, but not G-rectifiable, it gives rise to a counterexample to the Linearization Problem and the Equivariant Cancellation Problem (See section 6).

2. Preliminaries

Let $X = \operatorname{Spec} B$ be an affine variety with an algebraic G-action. Throughout this article, G is a reductive algebraic group. Then the affine domain B has a G-action and decomposes to $B = \oplus_{\omega \in \Omega} B_\omega$ where Ω is the set of isomorphism classes of irreducible G-modules and B_ω is the isotypic component of type ω. Each B_ω is a finitely generated module over B^G. Let δ be a G-equivariant k-derivation on B. Then it follows that $\delta(B_\omega) \subset B_\omega$. In particular, δ restricts to a k-derivation $\delta|_{B^G}$ on B^G. If δ is locally nilpotent, then $\delta|_{B^G}$ is locally nilpotent as well. The kernel B^δ of δ has a G-action. It holds that $(B^\delta)^G = (B^G)^\delta$. The restriction $\delta|_{B_\omega}$ is a B^G-module derivation on B_ω, i.e., $\delta(b_1 + b_2) = \delta(b_1) + \delta(b_2)$ for $b_1, b_2 \in B_\omega$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for $a \in B^G$ and

$b \in B_\omega$. We denote by $\mathrm{Der}(B)$ the B-module of derivations on B and by $\mathrm{Der}^G(B)$ the B^G-module of G-equivariant derivations on B.

Suppose that δ is locally nilpotent. The G_a-action on X induced by δ is fixed-point free if and only if the ideal of B generated by the image $\delta(B)$ contains 1. Suppose further that δ has a slice $s \in B$, i.e., $\delta(s) = 1$. Then the G_a-action induced by δ is fixed-point free. Note that we may assume that $s \in B^G$ since δ is G-equivariant. Then s is a slice of $\delta|_{B^G}$ as well.

We recall some properties of locally nilpotent derivations.

Lemma 2.1. *Let δ be a locally nilpotent derivation on B.*

(1) (Freudenburg [3]) *Suppose that there exist $b_1, \ldots, b_n \in B$ such that $\delta(b_i) \in b_{\sigma(i)}B$ for each i where σ is a permutation on n letters. Then in each orbit of σ, there is an i such that $\delta(b_i) = 0$. In particular, if $\delta(b) \in bB$, then $\delta(b) = 0$.*

(2) (cf. [6]) *Suppose that δ has a slice $s \in B$. Then s is transcendental over B^δ, and $B = B^\delta[s]$. The kernel B^δ is given by the image of the Dixmier map $\phi_s : B \to B$, which is the algebra homomorphism defined by*

$$\phi_s(b) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \delta^i(b) s^i \qquad \textit{for} \quad b \in B.$$

Then the kernel of ϕ_s is the ideal $(s) \subset B$. Suppose further that δ is G-equivariant. Then δ has a G-invariant slice $s_0 \in B^G$. It follows that $B^G = (B^G)^\delta[s_0]$ and the Dixmier map ϕ_{s_0} is G-equivariant.

Let $\pi : X \to X/\!/G$ be the quotient morphism which is defined by the inclusion $B^G \hookrightarrow B$. Then π induces an embedding of the fixed point locus X^G into $X/\!/G$. The G-action on X is called *fix-pointed* if $\pi|_{X^G}$ is an isomorphism, i.e., every closed orbit of X is a fixed point. It is known by Bass and Haboush [2] that if X is smooth and the G-action on X is fix-pointed then $X \cong X^G \times V$ for a G-representation space V. Hence if $X^G \cong X/\!/G$ is the affine space, then X is isomorphic to the affine space, say, $\mathbb{A}^n$ and the G-action on X is linearizable, i.e., isomorphic to a linear action in the automorphisms of $\mathbb{A}^n$.

By the slice theorem (Luna [5]), there is a finite stratification of $X/\!/G = \cup_i Y_i$ into locally closed subvarieties Y_i such that the isotropy groups of closed orbits in $\pi^{-1}(Y_i)$ are all conjugate to a fixed reductive subgroup H_i. Furthermore, if $\overline{Y_i} \supset Y_j$ then H_i is conjugate to a subgroup of H_j. The unique open dense stratum of $X/\!/G$ is called the principal stratum and the corresponding isotropy group a principal isotropy group.

Let us consider the case where G is abelian. Then it follows that $B = \oplus_{\chi \in \Omega} B_\chi$ where Ω is the set of characters of G and $B_\chi = \{b \in B \mid g \cdot b = \chi(g)b \text{ for all } g \in G\}$. An element of B_χ is called semi-invariant of weight χ. A derivation δ on $B = \oplus_{\chi \in \Omega} B_\chi$ is called semi-invariant of weight ω if $\delta(B_\chi) \subset B_{\chi+\omega}$ for every χ. Suppose that a locally nilpotent derivation δ on B which is semi-invariant of weight $-\omega$ has a slice $s \in B$. Then we may assume that s is semi-invariant of weight ω.

When G is a finite group, then B is a finite G-module over B^G. We consider the case where B is a finite module over a subring A of B. Let K (resp. L) be the quotient field of A (resp. B). Then L is a simple extension of K and L is written as $L = K(\theta)$ for some $\theta \in L$. Let δ be a derivation on A. Then δ extends uniquely to a derivation δ_K on K. Let $F(X)$ be the minimal polynomial of θ over K. Then it is well-known that a derivation δ_K lifts uniquely to a derivation Δ on L such that $\Delta(\theta) = -F^{\delta_K}(\theta)/F'(\theta)$, where $F^{\delta_K}(X)$ is the polynomial with all the coefficients of $F(X)$ replaced by their δ_K-images. The derivation Δ on L does not necessarily restrict to a derivation on B, i.e., $\Delta(B) \subset B$.

As for the derivation Δ, the following is known.

Theorem 2.2. *Let B be a normal affine domain and let A be a subring of B such that B is a finite A-module. Let δ be a derivation on A and let Δ be the derivation on the quotient field L of B such that $\Delta|_A = \delta$. Then the following assertions hold.*

(1) (Vasconcelos [10]) *If $\Delta(B) \subset B$ is satisfied, then Δ is locally nilpotent provided so is δ.*

(2) ([8]) *Suppose that δ is locally nilpotent. Suppose further that there exists a nonzero ideal $\mathfrak{a}$ of A satisfying the conditions :*
   (i) *The ideal $\mathfrak{a}$ has height at least two.*
   (ii) *The quotient morphism $\pi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is étale outside $V(\mathfrak{a})$.*

   Then $\Delta(B) \subset B$ and Δ is locally nilpotent.

(3) (Scheja-Storch [9]) *Let $\mathfrak{R}$ be the radical of the annihilator $\mathrm{Ann}(\Omega_{B/A})$ and let $\mathfrak{b} = A \cap \mathfrak{R}$. Then $\Delta(B) \subset B$ if and only if $\delta(\mathfrak{p}) \subset \mathfrak{p}$ for every height 1 prime divisor $\mathfrak{p}$ of $\mathfrak{b}$. In particular, if $\Omega_{B/A} = (0)$, i.e., B is unramified over A, then Δ satisfies $\Delta(B) \subset B$, i.e., δ lifts to a derivation Δ on B.*

If a derivation Δ on B restricts to a derivation δ on A, we call Δ a lifting of δ. When $A = B^G$ with G finite, the lifting Δ is G-equivariant (cf. Lemma 4.1).

A G-equivariant derivation Δ on B which restricts to a derivation δ on B^G is called a lifting of δ.

3. Equivariant derivations on G-modules

In this section, we observe G-equivariant derivations on G-modules.

Let W be a G-representation space of dimension n with the representation $\rho : G \to GL(W)$ and let $x_1, \cdots, x_n$ be the coordinate functions of W. The coordinate ring B of W is $B = k[x_1, \ldots, x_n]$ with the G-action given by $g \cdot (x_1, \ldots, x_n) = (x_1, \ldots, x_n)\rho^*(g)$ for $g \in G$ where $\rho^*(g) = {}^t\rho(g^{-1})$. The G-module $W^* = kx_1 + \cdots + kx_n$ is the dual module of W. The coordinate ring B has a decomposition $B = \oplus_{d\geq 0} S^d(W^*)$ where $S^d(W^*)$ is the d-th symmetric algebra of W^*. Each $S^d(W^*)$ is a G-module and decomposes to isotypic components.

Let δ be a derivation on B of a form

$$\delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$$

where $f_i \in B$. We decompose f_i to $f_i = \sum_{j=0}^r f_{ij}$ for $1 \leq i \leq n$ where $f_{ij} \in S^j(W^*)$ is a homogeneous polynomial of degree j. Then it follows that $\delta = \sum_{j=0}^r \delta_j$ where $\delta_j = f_{1j}\partial_{x_1} + \cdots + f_{nj}\partial_{x_n}$ is a homogeneous derivation on B. Let $V_j = kf_{1j} + \cdots + kf_{nj}$ be the submodule of $S^j(W^*)$.

Lemma 3.1. (1) *The derivation δ is G-equivariant if and only if*

$$g \cdot (f_{1i}, \ldots, f_{ni}) = (f_{1i}, \ldots, f_{ni})\rho^*(g)$$

for all $g \in G$ and $0 \leq i \leq r$.

(2) *Suppose that δ is G-equivariant. Then the G-module V_i is isomorphic to a G-submodule of W^*. Hence if W is irreducible, then $V_i = \{0\}$ or $V_i \cong W^*$.*

Proof. (1) The derivation δ is G-equivariant if and only if $(g\cdot\delta)(x_i) = (\delta \cdot g)(x_i)$ for every i and all $g \in G$, which is equivalent to that $g \cdot (f_1, \ldots, f_n) = (f_1, \ldots, f_n)\rho^*(g)$ for all $g \in G$. Since the G-action preserves the degree, the assertion follows.

(2) The G-module V_i is a G-submodule of $S^i(W^*)$. Let $q_i : W^* \to V_i$ be the surjection defined by $q_i(x_j) = f_{ji}$ for $1 \leq j \leq n$. Then by (1), q_i is G-equivariant. Hence the assertion follows. □

Suppose that W is irreducible. Let $\omega \in \Omega$ be the isomorphism class of W^*. By Lemma 3.1, it follows that for $\delta \in \mathrm{Der}^G(B)$, $k\delta(x_1) + \cdots + k\delta(x_n) = \sum_i V_i$ is a G-submodule of B_ω.

In the following in this section, we assume that δ is G-equivariant.

We consider the case where $\delta = \delta_0$. Then $\delta = c_1\partial_{x_1} + \cdots + c_n\partial_{x_n}$ for $c_i \in k$. Since δ is G-equivariant, it follows that ${}^t(c_1, \ldots, c_n)$ is the eigenvector with value 1 of $\rho(g)$ for all $g \in G$. Hence ${}^t(c_1, \ldots, c_n) \in W^G$ and we have the following.

Lemma 3.2. *If $\delta = c_1\partial_{x_1} + \cdots + c_n\partial_{x_n}$ is G-equivariant where $c_i \in k$ for $1 \le i \le n$, then ${}^t(c_1, \ldots, c_n) \in W^G$. As a consequence, if $W^G = \{0\}$, then δ is trivial. In particular, if W is an irreducible G-module, then $\delta = 0$.*

By Lemma 3.2, we have the following.

Proposition 3.3. *Suppose that $W^G = \{0\}$. Then there exists no G-equivariant G_a-action on W which is fixed-point free.*

Proof. Assume that there were a G-equivariant, fixed-point free, G_a-action on W. Let $\delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$ be the G-equivariant locally nilpotent derivation on B corresponding to the G_a-action. Since the G_a-action is fixed-point free, it follows that $h_1 f_1 + \cdots + h_n f_n = 1$ for $h_i \in B$. Decompose $\delta = \sum_{j=0}^{r} \delta_j$ where δ_j is a homogeneous derivation. Then by Lemma 3.2, δ_0 is trivial, i.e., each degree of f_i is positive. However, this contradicts to that $h_1 f_1 + \cdots + h_n f_n = 1$. Hence the assertion follows. □

Next, consider the case where $\delta = \delta_1$, i.e., $\delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$ where each $f_i \in B$ is linear in $x_1, \ldots, x_n$. Then by Lemma 3.1, it follows that the G-module $V_1 = kf_1 + \cdots + kf_n$ is isomorphic to a G-submodule of W^*.

Lemma 3.4. *Suppose that $\delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$ is a G-equivariant derivation on B where each $f_i \in B$ is linear in $x_1, \ldots, x_n$. If W is irreducible, then δ is of a form*

$$\delta = c(x_1\partial_{x_1} + \cdots + x_n\partial_{x_n})$$

for $c \in k$.

Proof. Suppose that δ is non-trivial and write $f_i = \sum_{j=1}^{n} x_j c_{ji}$ for $c_{ji} \in k$. By Lemma 3.1, it follows that $V_1 = kf_1 + \cdots + kf_n = W^*$. The $n \times n$ matrix $C = (c_{ji})$ gives a G-equivariant endomorphism of W^*. Since W^* is irreducible, C is a scalar matrix. Hence the assertion follows. □

We observe G-equivariant derivations on B when the dimension of $W//G$ is small.

First, we consider the case $\dim W//G = 0$, i.e., $B^G = k$. Then every isotypic component B_σ is a finite G-module over k and isomorphic to a direct sum of some copies of V where V is an irreducible G-module such that $\sigma = [V]$. Hence if W is irreducible, then $\mathrm{Der}^G(B)$ is a finite free module over k.

Lemma 3.5. *Let W be an irreducible G-module with* $\dim W//G = 0$ *and let δ be a G-equivariant derivation on B. Suppose that the multiplicity of W^* in the G-module B_ω for $\omega = [W^*]$ is one. Then δ is of a form*

$$\delta = c(x_1\partial_{x_1} + \cdots + x_n\partial_{x_n})$$

for $c \in k$. If δ is locally nilpotent, then δ is trivial.

Proof. Since the multiplicity of W^* in B_ω is one, it follows that $B_\omega = W^* = kx_1 + \cdots + kx_n$. Hence the assertion follows from Lemma 3.1. If δ is locally nilpotent, then δ is trivial by Lemma 2.1 (1). □

Example 3.1 Let $G = SL_n$ for $n \geq 2$ and let $W = R_1$ be the standard SL_n-representation space. Let $B = k[x_1, \ldots, x_n]$ be the coordinate ring of R_1. Then $B^G = k$ and B decomposes to $B = \oplus_{d\geq 0} S^d(R_1^*) = \oplus_{d\geq 0} R_d^*$ where $R_d^* = S^d(R_1^*)$ is an irreducible G-module. For $\omega = [R_1^*]$, $B_\omega = R_1^*$. Hence $\delta \in \mathrm{Der}^G(B)$ is of a form $\delta = c(x_1\partial_{x_1} + \cdots + x_n\partial_{x_n})$ for $c \in k$ and there is no non-trivial G-equivariant G_a-action on R_1.

Next, consider the case where $W//G$ is of one dimension. Then it follows that $W//G \cong \mathbb{A}^1$, i.e., $B^G = k[t]$ for some homogeneous polynomial $t \in B^G$. We have two cases. The first case is the case of the fix-pointed action, i.e., $W^G = k^1$ and $W = k^1 \oplus W'$ where W' is a G-submodule with the trivial quotient. In this case, we have $B = B'[t]$ where $B' = k[y_1, \ldots, y_{n-1}]$ is the coordinate ring of W' and the coordinate ring of W^G is $k[t]$. Then the $k[t]$-module $\mathrm{Der}^G(B)$ is a sum of $\mathrm{Der}(k[t])$ and $\mathrm{Der}^G(B') \otimes_k k[t]$. Hence every derivation on B^G lifts up to a G-equivariant derivation on B. The second case is that $W^G = \{0\}$. Then t is a homogeneous polynomial of degree $d \geq 2$. Since $B^G = k[t]$ is a principal ideal domain, it follows that $B_\omega = k[t] \otimes_k S$ where S is a free G-submodule of B. Let δ be a derivation on B^G. Then $\delta = f(t)(d/dt)$ for $f(t) \in k[t]$. If δ is locally nilpotent, then it follows that $\delta = c(d/dt)$ for $c \in k$.

Theorem 3.6. *Let* $\dim W//G = 1$ *and* $W^G = \{0\}$*. Then there exist no G_a-actions on W which restrict to non-trivial G_a-actions on $W//G$.*

Proof. As observed above, $B^G = k[t]$ with $d = \deg t > 1$ and a non-trivial locally nilpotent derivation on B^G is of a form $c(d/dt)$ for $c \in k^*$. Let Δ be a locally nilpotent derivation on B such that $\Delta|_{B^G} = c(d/dt)$. Write $\Delta = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n}$ for $f_i \in B$. Then $c = \Delta(t) = f_1(\partial_{x_1}t) + \cdots + f_n(\partial_{x_n}t)$. The degree of $\partial_{x_i}t$ is $d-1 > 0$ unless $\partial_{x_i}t = 0$, while the one of c is zero. This is a contradiction. Hence the assertion follows. □

Remark. There is no derivation on B which restricts to a locally nilpotent derivation on $B^G = k[t]$.

Example 3.2 Let G be a complex orthogonal group $O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. Let $B = \mathbb{C}[x, y]$ be the coordinate ring of the representation space W_n ($n \geq 1$), on which G acts by

$$\lambda \cdot (x, y) = (\lambda^n x, \lambda^{-n} y), \qquad \tau \cdot (x, y) = (y, x)$$

where $\lambda \in \mathbb{C}^*$ and τ is the generator of $\mathbb{Z}/2\mathbb{Z}$. Then $B^G = \mathbb{C}[t]$ where $t = xy$ and W_n has one-dimensional quotient. For $\omega = [W_n^*]$, $B_\omega = \mathbb{C}[t] \otimes_{\mathbb{C}} W_n^*$. Hence every G-equivariant derivation Δ on B is of a form

$$\Delta = f(t)(x\partial_x + y\partial_y)$$

for $f(t) \in B^G$. As a consequence, there is no non-trivial G-equivariant locally nilpotent derivation on B. Further, by the above remark, there is no derivation on B which restricts to a locally nilpotent derivation d/dt on B^G.

4. G-equivariant derivations with G finite

In this section, we observe G-equivariant derivations on G-modules with G finite.

Let G be a finite group and let W be a G-representation space. As remarked in Section 2, the coordinate ring B of W is a finite G-module over B^G. Let δ be a derivation on B^G. Then δ uniquely extends to a derivation δ_K on the quotient field K of B^G. Further, the derivation δ_K on K uniquely lifts up to a derivation Δ on the quotient field L of B. The following is verified.

Lemma 4.1. ([8])

(1) *Let Δ be a derivation on L. Then $\Delta(K) \subset K$ if and only if $g \cdot \Delta = \Delta \cdot g$ for every $g \in G$.*
(2) *Let Δ be a derivation on B. Then $\Delta(B^G) \subset B^G$ if and only if $g \cdot \Delta = \Delta \cdot g$ for every $g \in G$.*

By Lemma 4.1, if a derivation δ on B^G lifts up to a derivation Δ on B, then Δ is G-equivariant. By Theorem 2.2, the following holds.

Theorem 4.2. *Suppose that the following are satisfied.*

(1) *The principal isotropy group of W is trivial.*
(2) *The complement of the principal stratum has codimension ≥ 2 in W/G.*

Then every G_a-action on W/G lifts up to a G-equivariant G_a-action on W.

Example 4.1 Let $G = \mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n which acts on $B = \mathbb{C}[x, y]$ by

$$\sigma \cdot (x, y) = (\zeta x, \zeta^d y),$$

where σ is a generator of G, ζ is a primitive n-th root of unity and d is an integer such that $0 < d < n$ and $(d, n) = 1$. Then the isotropy group of every closed point of $W = \operatorname{Spec} B$ except the origin is trivial. The G-invariant subring B^G is generated by monomials $x^i y^j$ such that $i + dj \equiv 0 \pmod{n}$. The stratification of W/G is $W/G = (W/G - \{\bar{0}\}) \cup \{\bar{0}\}$ where $\bar{0}$ is the image of the origin of W by the quotient morphism. Since the principal isotropy group is trivial, it follows from Theorem 4.2 that every G_a-action on W/G lifts up to a G-equivariant G_a-action on W. The following is known.

Theorem 4.3. ([8]) *In the notation above, the following hold.*

(1) *Let $\Delta \in \operatorname{Der}^G(B)$. Then*

$$\Delta = (a_1 x + a_2 y^{d'})\partial_x + (b_1 x^d + b_2 y)\partial_y$$

where $a_1, a_2, b_1, b_2 \in B^G$ and d' is an integer such that $0 < d' < n$ and $dd' \equiv 1 \pmod{n}$.

(2) *Let δ' be an arbitrarily chosen, locally nilpotent derivation on B^G and let Δ' be the locally nilpotent derivation on B which lifts δ'. Then, after a suitable change of coordinates x, y of B, Δ' is given by $\Delta' = f(x^n)x^d\partial_y$ with $f(x^n) \in \mathbb{C}[x^n]$.*

Example 4.2 Let G be a dihedral group $D_d = \mathbb{Z}/d\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ for an odd prime integer d. Let $B = \mathbb{C}[x, y]$ be the coordinate ring of the representation space W_1, on which G acts by

$$\sigma \cdot (x, y) = (\zeta x, \zeta^{-1} y), \qquad \tau \cdot (x, y) = (y, x)$$

where σ is a generator of $\mathbb{Z}/d\mathbb{Z}$, τ the generator of $\mathbb{Z}/2\mathbb{Z}$, and ζ is a d-th primitive root of unity. Then $B^G = \mathbb{C}[s, t]$ where $s = x^d + y^d$ and $t = xy$. The stratification of $W_1/G \cong \mathbb{A}^2$ is

$$\mathbb{A}^2 = (\mathbb{A}^2 - V) \cup (V - \{\bar{0}\}) \cup \{\bar{0}\}$$

where $V = \{(s, t) \mid s^2 - 4t^d = 0\}$. The isotropy groups corresponding to the strata $\mathbb{A}^2 - V$, $V - \{\bar{0}\}$, $\{\bar{0}\}$ are the trivial group, $\mathbb{Z}/2\mathbb{Z}$ (the second factor), G, respectively. We have the following result.

Theorem 4.4. ([8])

(1) *Let* $\Delta \in \mathrm{Der}^G(B)$*. Then*

$$\Delta = f_1(x\partial_x + y\partial_y) + \sum_{i=1}^{l} f_{2i}(y^{id-1}\partial_x + x^{id-1}\partial_y)$$

where $f_1, f_{2i} \in B^G$ *and* $l \geq 1$*.*

(2) *There is no non-trivial* G*-equivariant locally nilpotent derivation on* B*.*

By Lemma 4.1 and Theorem 4.4, any G_a-action on $W_1/G \cong \mathbb{A}^2$ does not lift up to a G_a-action on W_1.

Example 4.3 Let G be the symmetric group S_n on n letters, which acts on the polynomial ring $B = \mathbb{C}[x_1, \ldots, x_n]$ in the standard way such that $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in S_n$. Then $B^G = \mathbb{C}[s_1, \ldots, s_n] = \mathbb{C}[t_1, \ldots, t_n]$, where s_i and t_i are the i-th elementary symmetric polynomials

$$s_i = \sum_{j_1 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}, \quad t_i = \sum_{j=1}^{n} x_j^i \ .$$

Let $W = \mathrm{Spec}\, B$. Then $W^G = \mathbb{C}$ and $W/G \cong \mathbb{A}^n$. The principal isotropy group is trivial and the complement of the principal stratum has codimension 1.

Theorem 4.5. ([8]) *The* B^G*-module* $\mathrm{Der}^G(B)$ *is freely generated by*

$$\Delta_i = x_1^i \partial_{x_1} + \cdots + x_n^i \partial_{x_n}$$

for $0 \leq i \leq n-1$*. If* $\Delta \in \mathrm{Der}^G(B)$ *is locally nilpotent, then* $\Delta = f\Delta_0$ *where* f *is an element of* B^G *such that* $\Delta_0(f) = 0$*.*

Let $\delta_0 = \Delta_0|_{B^G}$. Then the locally nilpotent derivation δ_0 on $B^G = \mathbb{C}[t_1, \cdots, t_n]$ has a slice $s = t_1/n$. By Theorem 4.5, a G_a-action on $W/G \cong \mathbb{A}^n$ is liftable onto W if and only if it is given by the locally nilpotent derivation $f\delta_0$ for $f \in (B^G)^{\delta_0}$.

5. Equivariant derivations on affine G-domains

In this section, we observe G-equivariant G_a-actions on smooth complex affine G-varieties when the dimension of the algebraic quotient is small.

Let $X = \mathrm{Spec}\, B$ be a smooth complex affine G-variety. Let Δ be a non-trivial G-equivariant locally nilpotent derivation on B. Then Δ restricts to a locally nilpotent derivation $\delta = \Delta|_{B^G}$ on B^G. Note that $(B^\Delta)^G = (B^G)^\delta$.

Suppose that $\dim X//G = 0$, i.e., $B^G = \mathbb{C}$. Then it follows from [2] that X is isomorphic to a G-representation space W. Hence this case reduces to the case where X is a G-representation space.

Suppose that $\dim X//G = 1$. If X is acyclic, then it follows that $B^G = \mathbb{C}[t]$ for some $t \in B^G$ (cf. [4]). The locally nilpotent derivation δ on B^G is of a form $\delta = c(d/dt)$ for $c \in \mathbb{C}$. The G-equivariant G_a-action on X given by Δ induces a non-trivial G_a-action on $X//G \cong \mathbb{A}^1$ if and only if Δ has a slice.

Theorem 5.1. *Suppose that $X = \operatorname{Spec} B$ is a smooth, acyclic complex affine G-variety with one-dimensional quotient. Then X is isomorphic to a G-representation space $W \oplus \mathbb{C}$ if and only if there exist a G-equivariant G_a-action on X which induces a non-trivial G_a-action on $X//G$. Here, W is a G-representation space such that $W//G$ is a point and $\mathbb{C}$ is a one-dimensional trivial G-representation space.*

Proof. "Only if" part is obvious. We show that "if" part. As observed above, there is a G-equivariant locally nilpotent derivation Δ on B having a slice $t \in B^G$ and $B = B^\Delta[t]$. Let $Y = \operatorname{Spec} B^\Delta$. Then Y is a smooth, acyclic G-variety since $X \cong Y \times \mathbb{A}^1$. Since $(B^\Delta)^G = (B^G)^\delta = \mathbb{C}$, it follows from [2] that Y is isomorphic to a G-representation space W such that $W//G$ is a point. Hence $X \cong Y \times \mathbb{A}^1 \cong W \oplus \mathbb{C}$ and the assertion follows. □

Suppose that $\dim X//G = 2$. If Δ has a slice, then $Y = \operatorname{Spec} B^\Delta$ has one-dimensional quotient since $\dim (B^\Delta)^G = \dim (B^G)^\delta = 1$.

Theorem 5.2. *Suppose that $X = \operatorname{Spec} B$ is a smooth, acyclic complex affine G-variety with two-dimensional quotient. Then X is isomorphic to a G-representation space $W \oplus \mathbb{C}^2$ if and only if there exists a locally nilpotent derivation on B^G having a slice and every G_a-action on $X//G$ lifts up to a G-equivariant G_a-action on X. Here, W is a G-representation space such that $W//G$ is a point.*

Proof. We show that "if" part. Let δ be a locally nilpotent derivation on B^G with a slice $s \in B^G$. Then δ lifts up to a G-equivariant locally nilpotent derivation Δ on B. Since Δ has a slice $s \in B^G$, it follows that $B = B^\Delta[s]$. Hence $Y = \operatorname{Spec} B^\Delta$ is a smooth, acyclic G-variety with one-dimensional quotient. Thus $(B^\Delta)^G = \mathbb{C}[t]$ for $t \in B^\Delta$, and $B^G = \mathbb{C}[t, s]$. The locally nilpotent derivation d/dt on B^G lifts up to a G-equivariant locally nilpotent derivation Δ' on B. Since $\Delta'(s) = ds/dt = 0$, Δ' induces a G-equivariant G_a-action on the G-subvariety Y of X. It follows from Theorem 5.1 that $Y \cong W \oplus \mathbb{C}$ for a G-representation space W such that $\dim W//G = 0$. Hence $X \cong Y \times \mathbb{A}^1 \cong W \oplus \mathbb{C}^2$. □

6. Semi-invariant locally nilpotent derivations associated to equivariant embeddings

In this section, we observe semi-invariant locally nilpotent derivations associated with G-equivariant embeddings where G is abelian. Throughout this section, G is an abelian reductive algebraic group.

First, we make some observations on G-equivariant embeddings. Let V and $\tilde{V}$ be G-representation spaces, respectively. Since G is abelian, V (resp. $\tilde{V}$) is a direct sum of one-dimensional G-representation spaces. The coordinate ring S (resp. R) of V (resp. $\tilde{V}$) is a polynomial ring with semi-invariant coordinates. Let $\varphi : V \hookrightarrow \tilde{V}$ be a G-equivariant embedding of G-varieties, which we abbreviate to a G-embedding. We may assume that φ maps the origin of V to the origin of $\tilde{V}$. Since φ induces a G-equivariant injective homomorphism of the tangent space at the origin of V into the tangent space at the origin of $\tilde{V}$, it follows that $\tilde{V} = V \oplus V'$ for some G-representation space V'. Let $S = k[v_1, \ldots, v_m]$ and $R = k[X_1, \ldots, X_n]$ where v_i $(1 \leq i \leq m)$ is semi-invariant of weight α_i and X_j $(1 \leq j \leq n)$ of weight β_j. Then we may and assume $\beta_i = \alpha_i$ for $1 \leq i \leq m$. For two G-embeddings φ and φ' of V into $\tilde{V}$, φ' is called G-equivalent to φ if there is a G-equivariant automorphism γ of $\tilde{V}$ such that $\varphi' = \gamma \circ \varphi$. A G-embedding $\varphi : V \to \tilde{V}$ is called G-*rectifiable* if there exists a system of semi-invariant coordinate functions $f_1, \ldots, f_n$ on $\tilde{V}$ such that the image $\varphi(V)$ is defined by the G-stable ideal $(f_{m+1}, \ldots, f_n)$. If $\varphi : V \to \tilde{V}$ is G-equivalent to the standard G-embedding $V \hookrightarrow V \oplus V' = \tilde{V}$, then φ is G-rectifiable. Forgetting the G-action, a G-rectifiable embedding $V \to \tilde{V}$ is a rectifiable embedding $\mathbb{A}^m \to \mathbb{A}^n$.

Now, we construct semi-invariant locally nilpotent derivations associated with $\varphi : V \to \tilde{V}$. Let B be a polynomial ring with a linear G-action

$$B = k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]$$

where the weight of u, v_i $(1 \leq i \leq m)$, x_j $(1 \leq j \leq n)$ is 0, α_i, β_j, respectively. As a G-variety, $X = \operatorname{Spec} B$ is isomorphic to $L \oplus V \oplus \tilde{V}$ where L is the trivial G-representation space of one dimension. We give B a $\mathbb{Z}$-grading by

$$\deg u = -1, \quad \deg v_i = 0, \quad \deg x_j = d$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$ where d is a fixed positive integer. There is a bijective correspondence between algebraic actions of an algebraic torus G_m on $\operatorname{Spec} B$ and $\mathbb{Z}$-graded k-algebra structures on B. Hence there is a G_m-action on X corresponding to this grading, which we call the T-action where $T = G_m$. The T-action on X commutes with the G-action. Note that

$$B^T = k[v_1, \dots, v_m, u^d x_1, \dots, u^d x_n]$$

and the algebraic quotient $X//T = \operatorname{Spec} B^T$ is G-equivariantly isomorphic to $V \oplus \tilde{V}$. It follows that $B = \oplus_{(\chi,i)\in\Omega\oplus\mathbb{Z}} B_{\chi,i}$ where $B_{\chi,i} = \{b \in B \mid (g,t)\cdot b = t^i \chi(g) b \quad \text{for all } (g,t) \in G \times T\}$. The weight of u, v_i $(1 \le i \le m)$, x_j $(1 \le j \le n)$ is $(0,-1)$, $(\alpha_i, 0)$, (β_j, d), respectively. As a $(G \times T)$-representation space,

$$X = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d}$$

where $L = \operatorname{Spec} k[u]$, $V_0 = \operatorname{Spec} S$, $V_{-d} = \operatorname{Spec} k[x_1, \dots, x_m]$, and $V'_{-d} = \operatorname{Spec} k[x_{m+1}, \dots, x_n]$. Note that $V_0 = V = V_{-d}$ as a G-representation space. Let $\Phi : R \to S$ be the G-equivariant surjection associated with the G-embedding $\varphi : V \to \tilde{V}$. Let f_j be the image of X_j by $\Phi : R \to S$ for $1 \le j \le n$. We define a derivation δ_i on B by

$$\delta_i = u^d \partial_{v_i} + (\partial_{v_i} f_1)\partial_{x_1} + \cdots + (\partial_{v_i} f_n)\partial_{x_n}$$

for $1 \le i \le m$. Then δ_i is locally nilpotent and semi-invariant of weight $(-\alpha_i, -d)$, i.e., $\delta_i(B_{\chi,j}) \subset B_{\chi-\alpha_i, j-d}$ for every $(\chi, j) \in \Omega \oplus \mathbb{Z}$ and $\delta_i\delta_l = \delta_l\delta_i$ for any i and l. We consider the sequence $\Delta_\varphi = (\delta_1, \dots, \delta_m)$ of mutually commuting semi-invariant locally nilpotent derivations δ_i's defined above. The sequence $\Delta_\varphi = (\delta_1, \dots, \delta_m)$ has a semi-invariant slice system $(s_1, \dots, s_m)$, i.e., each δ_i has a semi-invariant slice $s_i \in B$ such that $\delta_i(s_l) = 0$ for $i \ne l$. In fact, since Φ is a surjection, there exists a polynomial $F_i \in k[X_1, \dots, X_n]$ such that

$$F_i(f_1, \dots, f_n) = v_i$$

for $1 \le i \le m$. Let $s_i \in B$ be an element satisfying

$$F_i(f_1 - u^d x_1, \dots, f_n - u^d x_n) = v_i - u^d s_i.$$

Since $\delta_l(f_j - u^d x_j) = 0$ for $1 \le l \le m$ and $1 \le j \le n$, it follows that $\delta_l(v_i - u^d s_i) = 0$, i.e., $\delta_i(s_i) = 1$ and $\delta_l(s_i) = 0$ for $i \ne l$. We can take s_i to be semi-invariant of weight (α_i, d). Hence $(s_1, \dots, s_m)$ is a semi-invariant slice system of Δ_φ. We call Δ_φ the sequence associated to the G-embedding φ. In the following, a sequence $\Delta = (\delta_1, \dots, \delta_m)$ implies a sequence of mutually commuting locally nilpotent derivations δ_i's on B such that each δ_i is of the form

$$\delta_i = u^d \partial_{v_i} + f_{i1}\partial_{x_1} + \cdots + f_{in}\partial_{x_n} \qquad (*)$$

with $f_{ij} \in B^T$ and semi-invariant of weight $(-\alpha_i, -d)$ for $1 \le i \le m$.

Conversely, for a given sequence $\Delta = (\delta_1, \dots, \delta_m)$ having a slice system, there exists a G-embedding $\varphi_\Delta : V \to \tilde{V}$ associated to Δ ([7]).

Proposition 6.1. ([7], cf. [6]) *Let $\varphi : V \to \tilde{V}$ be a G-embedding and let $\Delta_\varphi = (\delta_1, \ldots, \delta_m)$ be the sequence associated to φ. Then the G-embedding φ_{Δ_φ} associated to Δ_φ is G-equivalent to φ.*

For two sequences $\Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)})$ and $\Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)})$, we say that Δ_1 and Δ_2 are G-equivalent if there is a $(G \times T)$-equivariant $k[v_1, \ldots, v_m]$-automorphism ψ of B which satisfies $\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)}$ for every i. The following is known.

Theorem 6.2. ([7], cf. [6]) *There is a bijective correspondence between the G-equivalence classes of G-embeddings $V \to \tilde{V}$ and the G-equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems.*

Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence having a semi-invariant slice system $(s_1, \ldots, s_m)$. Let

$$A := \cap_{i=1}^m B^{\delta_i}.$$

Then A inherits the $(G \times T)$-action on B. Let $Y = \operatorname{Spec} A$. Since Δ has a semi-invariant slice system $(s_1, \ldots, s_m)$, it follows from Lemma 2.1 that $B = A[s_1, \cdots, s_m]$, i.e.,

$$Y \times V_{-d} \cong X = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d}. \qquad (1)$$

Note that Y is smooth. If Y is isomorphic to a $(G \times T)$-representation space, then it follows from (1) that Y is necessarily isomorphic to the $(G \times T)$-representation space

$$W := L \oplus V_0 \oplus V'_{-d}.$$

Here, we recall that a locally nilpotent derivation having a slice induces an algebraic G_m-action. For $1 \le i \le m$, δ_i and its slice s_i induce an algebraic G_m-action on X, which we call the T_i-action where $T_i = G_m$ for $1 \le i \le m$. The T_i-action on X corresponds to the $\mathbb{Z}$-grading on $B = A[s_1, \ldots, s_m]$ such that $\deg s_i = 1$ and $\deg a = 0$ for $a \in B^{\delta_i} - \{0\}$. The subalgebra B^{T_i} is equal to $B^{\delta_i} \cong B/(s_i)$. Since the T_i-actions commute with each other, the m-dimensional torus $T' = T_1 \times \cdots \times T_m$ acts on X and $B^{T'} = A$. Since the $(G \times T)$-action commutes with every T_i-action, $G \times T \times T'$ acts on X. Note that $Y = X/\!/T'$ since $A = B^{T'}$. The algebraic quotient $X/\!/T' = \operatorname{Spec} B^{T'}$ is $(G \times T)$-equivariantly isomorphic to the fixed-point locus $X^{T'} = \operatorname{Spec} B/(s_1, \ldots, s_m)$.

The following is verified.

Theorem 6.3. ([7]) *Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence having a slice system. Then the following are equivalent.*

(1) *Y is isomorphic to the $(G \times T)$-representation space W.*

(2) *A is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action.*
(3) *The G-embedding $\varphi_\Delta : V \to \tilde{V}$ associated to Δ is G-rectifiable.*
(4) *The $(G \times T \times T')$-action on X is linearizable.*

The case that G is trivial is shown in [6] (cf. [1]). If the associated G-embedding φ_Δ is G-rectifiable, then it follows from Theorem 6.3 that $Y \cong W$, namely, the equivariant cancellation holds.

Suppose that there is a G-embedding $\varphi : V \to \tilde{V}$ which is rectifiable, but not G-rectifiable. Let $\Delta_\varphi = (\delta_1, \dots, \delta_m)$ be the sequence associated to φ and let $Y = \operatorname{Spec} A$ where $A = \cap_{i=1}^m B^{\delta_i}$. Then Y is isomorphic to a T-representation space, hence the affine space of dimension $n+1$, but not isomorphic to a $(G \times T)$-representation space. Namely, the $(G \times T)$-action on $Y \cong \mathbb{A}^{n+1}$ is non-linearizable.

Note that W is isomorphic to $L \oplus \tilde{V}$ as a G-representation space.

Proposition 6.4. ([7]) *Let Δ be a sequence having a slice system. Suppose that the G-action on X is fix-pointed and $V^G = \{0\}$. Then $Y \cong L \oplus \tilde{V}$ as a G-representation space.*

Proof. Since the G-action on Y is fix-pointed as well, it follows that $Y/\!/G \cong Y^G$. Since $V^G = \{0\}$, it follows from (1) that $Y^G \cong L \oplus (V'_{-d})^G$, hence $Y/\!/G$ is an affine space. Therefore Y is isomorphic to a G-representation space, hence $Y \cong L \oplus \tilde{V}$. □

Remark. Under the assumption in Proposition 6.4, the G-action on $Y_i = \operatorname{Spec} A_i$ is fix-pointed where $A_i = B^{\delta_i}$. Hence it follows that every Y_i is isomorphic to a G-representation space.

Suppose that G is an r-dimensional torus $(k^*)^r$. Then B is $\mathbb{Z}^r$-graded. If every component of $\deg v_i$ is positive and every component of $\deg x_j$ is non-negative, then the G-action on X is fix-pointed and $V^G = \{0\}$. By Proposition 6.4, it follows that $Y \cong L \oplus \tilde{V}$.

References

[1] T. Asanuma, Non-linearizable algebraic k^*-actions on affine spaces, Invent. Math. **138** (1999), 281–306.
[2] H. Bass and W. Haboush, Linearizing certain reductive group actions, Trans. Amer. Math. Soc. **292** (1985), 463–482.
[3] G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia Math. Sciences, **136**, Invariant theory and algebraic transformation groups VII, Springer Verlag, Berlin 2006.
[4] H. Kraft and G. Schwarz, Reductive group actions with one-dimensional quotient, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 1–97.

[5] D. Luna, Slice etales, Bull. Soc. Math. France, Memoire 33 (1973), 81–105.
[6] K. Masuda, Homogeneous locally nilpotent derivations having slices and embeddings of affine spaces, J. Algebra **321** (2009), 1719–1733.
[7] K. Masuda, Equivariant embeddings and semi-invariant locally nilpotent derivations, preprint.
[8] K. Masuda and M. Miyanishi, Lifting of the additive group scheme actions, Tohoku Math. J. **61** (2009), 267–286.
[9] G. Scheja and U. Storch, Fortsetzung von Derivationen, J. Algebra **54** (1978), 353–365.
[10] W. V. Vasconcelos, Derivations of commutative noetherian rings, Math. Zeit. **112** (1969), 229–233.

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