Let $G$ be an abelian reductive algebraic group defined over a field of characteristic 0. Let $V$ and $\tilde{V}$ be $G$-representation spaces. We assume that $\tilde{V}$ is a direct sum of one-dimensional $G$-representation spaces. We show that up to the $G$-equivalence, $G$-equivariant embeddings $V \to \tilde{V}$ bijectively correspond to the sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems where $m = \dim V$ and $\delta_i$'s are semi-invariant, mutually commuting locally nilpotent derivations of some form on a polynomial ring $B$ with a linear $(G \times G_m)$-action. For a sequence $\Delta$ as above, the intersection $B^\Delta$ of the kernels of $\delta_i$ for $1 \leq i \leq m$ inherits the $(G \times G_m)$-action on $B$. We show that $B^\Delta$ is a polynomial ring with semi-invariant coordinates if and only if the $G$-equivariant embedding associated to $\Delta$ is equivariantly rectifiable. Our results are the equivariant extension of [9].

1. Introduction and results

Let $k$ be a field of characteristic 0, which is the ground field. Let $G$ be an abelian reductive algebraic group. A $G$-representation space, abbreviated to a $G$-representation, is an affine space with a linear $G$-action as a $G$-variety. Let $V$ and $\tilde{V}$ be $G$-representations of dimension $m$ and $n$, respectively. We assume that $\tilde{V}$ is a direct sum of one-dimensional $G$-representations. Hence the coordinate ring of $\tilde{V}$ is a polynomial ring with semi-invariant coordinates. Let $\varphi : V \hookrightarrow \tilde{V}$ be a $G$-equivariant embedding of $G$-varieties, which we abbreviate to a $G$-embedding. We may assume that $\varphi$ maps the origin of $V$ to the origin of $\tilde{V}$. Since $\varphi$ induces a $G$-equivariant injective homomorphism of the tangent space at the origin of $V$ into the tangent space at the origin of $\tilde{V}$. Since $\varphi$ induces a $G$-equivariant injective homomorphism of the tangent space at the origin of $V$ into the tangent space at the origin of $\tilde{V}$. Since $\varphi$ induces a $G$-equivariant injective homomorphism of the tangent space at the origin of $V$ into the tangent space at the origin of $\tilde{V}$.
\( \tilde{V} \), it follows that \( \tilde{V} = V \oplus V' \) for some \( G \)-representation \( V' \). Hence \( V \) is a direct sum of one-dimensional \( G \)-representations as well. For two \( G \)-embeddings \( \varphi \) and \( \varphi' \) of \( V \) into \( \tilde{V} \), \( \varphi' \) is called \( G \)-equivalent to \( \varphi \) if there is a \( G \)-equivariant automorphism \( \gamma \) of \( \tilde{V} \) such that \( \varphi' = \gamma \circ \varphi \). A \( G \)-embedding \( \varphi : V \to \tilde{V} \) is called \( G \)-rectifiable if there exists a system of semi-invariant coordinate functions \( f_1, \ldots, f_n \) on \( \tilde{V} \) such that the image \( \varphi(V) \) is defined by the \( G \)-stable ideal \( (f_{m+1}, \ldots, f_n) \). If \( \varphi : V \to \tilde{V} \) is \( G \)-equivariant to the standard \( G \)-embedding \( V \hookrightarrow V \oplus V' = \tilde{V} \), then \( \varphi \) is \( G \)-rectifiable. Forgetting the \( G \)-action, a \( G \)-rectifiable embedding \( \tilde{V} \to V \) is a rectifiable embedding \( \mathbb{A}^n \to \mathbb{A}^n \). It is known by van den Essen and van Rossum [5] that there is a sequence of locally nilpotent derivations associated to a given embedding \( \mathbb{A}^m \to \mathbb{A}^n \). In fact, the sequences of locally nilpotent derivations of some form bijectively correspond to embeddings up to the equivalence ([9]). In this article, we show that the same holds true equivariantly.

We fix the notation and state our results. Let \( \Omega \) be the set of characters of \( G \). Let \( V = \text{Spec } k[v_1, \ldots, v_m] \) and let \( v_i \) \( (1 \leq i \leq m) \) be a semi-invariant of weight \( \alpha_i \in \Omega \), i.e., \( g \cdot v_i = \alpha_i(g) v_i \) for all \( g \in G \). Let \( \beta_1, \ldots, \beta_u \in \Omega \) be the weights of a system of semi-invariant coordinate functions of \( \tilde{V} \). Let \( B \) be a polynomial ring \( k[u, v_1, \ldots, v_m, x_1, \ldots, x_n] \) with a linear \( G \)-action such that the weight of \( u, v_i \) \( (1 \leq i \leq m) \), \( x_j \) \( (1 \leq j \leq n) \) is 0, \( \alpha_i, \beta_j \), respectively. As a \( G \)-variety, \( Y = \text{Spec } B \) is isomorphic to \( L \oplus V \oplus \tilde{V} \) where \( L \) is the trivial \( G \)-representation of one dimension. We give \( B \) a \( \mathbb{Z} \)-grading by

\[
\deg u = -1, \quad \deg v_i = 0, \quad \deg x_j = d
\]

for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) where \( d \) is a fixed positive integer. There is an algebraic action of an algebraic torus \( G_m \) on \( Y \) corresponding to this grading, which we call the \( T \)-action where \( T = G_m \). The \( T \)-action on \( Y \) commutes with the \( G \)-action. Note that the algebraic quotient \( Y//T = \text{Spec } B^T \) is \( G \)-equivariantly isomorphic to \( V \oplus \tilde{V} \) where \( B^T \) is the \( T \)-invariants of \( B \). It follows that \( B = \bigoplus_{(\chi, i) \in \Omega \oplus \mathbb{Z}} B_{\chi,i} \) where

\[
B_{\chi,i} = \{ b \in B \mid (g, t) \cdot b = t^i \chi(g) b \text{ for all } (g, t) \in G \times T \}.
\]

A derivation \( \delta \) on \( B = \bigoplus_{(\chi, i) \in \Omega \oplus \mathbb{Z}} B_{\chi,i} \) is called semi-invariant of weight \( (\omega, \ell) \) if \( \delta(B_{\chi,i}) \subset B_{\chi+\omega,i+\ell} \) for every \( (\chi, i) \). We consider a sequence \( \Delta = (\delta_1, \ldots, \delta_m) \) of semi-invariant, locally nilpotent derivations \( \delta_i \)'s on \( B \) satisfying \( \delta_i \delta_l = \delta_l \delta_i \) for any \( i \) and \( l \). For such two sequences \( \Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)}) \) and \( \Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)}) \), we say that \( \Delta_1 \) and \( \Delta_2 \) are \( G \)-equivalent if there is a \( (G \times T) \)-equivariant \( k[v_1, \ldots, v_m] \)-automorphism \( \psi \) of \( B \) which satisfies \( \delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)} \) for every \( i \). We say that
the sequence $\Delta = (\delta_1, \ldots, \delta_m)$ of mutually commuting locally nilpotent derivations on $B$ has a slice system $(s_1, \ldots, s_m)$ if each $\delta_i$ has a slice $s_i \in B$, i.e., $\delta_i(s_i) = 1$, such that $\delta_i(s_l) = 0$ for $i \neq l$. When the $G$-action is trivial, it is shown in [9] that there is a bijective correspondence between the equivalence classes of embeddings $\mathbb{A}^m \to \mathbb{A}^n$ and the equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems where $\delta_i$'s are mutually commuting locally nilpotent derivations on $B$ and semi-invariant of weight $-d$ of a form

$$\delta_i = u^d \partial_{v_i} + f_{i1} \partial_{x_1} + \cdots + f_{in} \partial_{x_n}$$

for $f_{ij} \in B^T$. For a sequence $\Delta = (\delta_1, \ldots, \delta_m)$ as above, let $B^\Delta = \cap_{i=1}^m B^{\delta_i}$ where $B^{\delta_i}$ denotes the kernel of $\delta_i$. Then $B^\Delta$ inherits the $T$-action on $B$. It is known as well that $B^\Delta$ is a polynomial ring with semi-invariant coordinates with respect to the $T$-action if and only if the embedding $\varphi_\Delta : \mathbb{A}^m \to \mathbb{A}^n$ associated to $\Delta$ is rectifiable ([9]). Throughout this article, a sequence $\Delta = (\delta_1, \ldots, \delta_m)$ implies a sequence of mutually commuting locally nilpotent derivations $\delta_i$'s on $B$ such that each $\delta_i$ is of the form (*) and semi-invariant of weight $(-\alpha_i, -d)$ for $1 \leq i \leq m$ unless otherwise stated. The next result follows from Proposition 4.3 and Lemma 4.6, which appear below.

**Theorem 1.1.** There is a bijective correspondence between the $G$-equivalence classes of $G$-embeddings $V \to \tilde{V}$ and the $G$-equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems.

The kernel $B^\Delta$ of a sequence $\Delta$ inherits the $(G \times T)$-action on $B$. We show the following (Theorem 5.1).

**Theorem 1.2.** Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence having a slice system. Then $B^\Delta$ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action if and only if the $G$-embedding $\varphi_\Delta : V \to \tilde{V}$ associated to $\Delta$ is $G$-rectifiable.

Suppose that a $G$-embedding $\varphi : V \to \tilde{V}$ is rectifiable, but not $G$-rectifiable. Then $X = \text{Spec } B^\Delta$ is isomorphic to a $T$-representation, hence the affine space of dimension $n + 1$, but not isomorphic to a $(G \times T)$-representation. Namely, the $(G \times T)$-action on $X \cong \mathbb{A}^{n+1}$ is non-linearizable.

We observe the $G$-rectifiability of $G$-embeddings in section 3. In section 5, we observe an example of non-rectifiable $\mathbb{Z}/2\mathbb{Z}$-embedding and discuss problems related to it. In general, it is not easy to determine whether the kernel $B^\Delta$ of a sequence $\Delta$ having a slice system is a polynomial ring or not. However, there are some cases that $B^\Delta$ is a polynomial ring with semi-invariant coordinates, i.e., $X = \text{Spec } B^\Delta$. 
is isomorphic to a representation of $G$ or $G \times T$. We give a couple of examples. Suppose that the ground field $k$ is algebraically closed. The $G$-action on an affine variety $Y$ is called fix-pointed if every closed orbit of $Y$ is a fixed point. Suppose that the $G$-action on $\text{Spec } B$ is fix-pointed and $V^G = \{0\}$. Then $B^\Delta$ is a polynomial ring with semi-invariant coordinates with respect to the $G$-action (Proposition 5.4). Consider, next, the case where $G$ is a complex algebraic torus and $V$ and $\tilde{V}$ are complex $G$-representations. Suppose that $\dim \tilde{V}/G \leq 1$. Then it follows from the results of Bass and Haboush [2] and Kraft and Schwarz [8] that $B^\Delta$ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action (Theorem 5.5). Hence by Theorems 1.1 and 1.2, it follows that every $G$-embedding $V \rightarrow \tilde{V}$ is $G$-rectifiable in this case.

Our results hold for a reductive group $G$ not necessarily abelian under the assumption that $\tilde{V}$ is a direct sum of one-dimensional $G$-representations unless otherwise $G$ is specified.

Acknowledgements The author is grateful to the referee for making suggestions for improving the presentation.

2. Preliminaries

We collect some results on the kernel of a sequence of semi-invariant, mutually commuting locally nilpotent derivations on an affine domain with an action of an abelian reductive group.

Let $G$ be an abelian reductive algebraic group and let $B$ be an affine domain with an algebraic $G$-action. The subalgebra $B^G$ of $G$-invariants is finitely generated over $k$. Let $\Omega$ be the set of characters of $G$. We assume that $B$ has a decomposition $B = \bigoplus_{\chi \in \Omega} B_\chi$ where

$$B_\chi = \{ b \in B \mid g \cdot b = \chi(g)b \quad \text{for all } g \in G \}. $$

An element of $B_\chi$ is called semi-invariant of weight $\chi$.

Suppose that $\delta$ is a locally nilpotent derivation on $B$ which is semi-invariant of weight $-\omega$ and that $\delta$ has a slice $s \in B$. Then we may assume that $s$ is semi-invariant of weight $\omega$.

Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence of mutually commuting locally nilpotent derivations on $B = \bigoplus_{\chi \in \Omega} B_\chi$. Suppose that $\delta_i$ is semi-invariant of weight $-\omega_i$ for $1 \leq i \leq m$ and that $\Delta$ has a semi-invariant slice system $(s_1, \ldots, s_m)$ such that $s_i$ is of weight $\omega_i$.

Lemma 2.1. (cf. [9, Lemma 2.1]) Under the notation and assumption above, we have the following.
(1) The slice $s_i$ is transcendental over $B^{\delta_i}$ and $B = B^{\delta_i}[s_i]$. Hence

$$B = A[s_1, \ldots, s_m] \quad \text{for} \quad A = \bigcap_{i=1}^{m} B^{\delta_i}.$$

(2) Let $\pi_{s_i} : B \to B$ be the Dixmier map induced by $\delta_i$ and $s_i$, which is the algebra homomorphism defined by

$$\pi_{s_i}(b) = \sum_{j \geq 0} \frac{(-1)^j}{j!} \delta_i^j(b)s_i^j \quad \text{for} \quad b \in B.$$

Then

(i) $\pi_{s_i}(B_\chi) \subset B_\chi$ for every $\chi \in \Omega$.

(ii) $B^{\delta_i} = \pi_{s_i}(B)$.

(iii) The kernel of $\pi_{s_i}$ is the $G$-stable ideal $(s_i) \subset B$.

Hence $A = \pi_s(B)$ where $\pi_s = \pi_{s_m} \circ \cdots \circ \pi_{s_1}$ and $A$ inherits the $G$-action.

The sequence $\Delta = (\delta_1, \ldots, \delta_m)$ on $B$ having a semi-invariant slice system induces a $\mathbb{Z}^m$-grading on $B$, hence an algebraic action of an $m$-dimensional torus $(G_m)^m$ on $Y := \text{Spec } B$. In fact, for $1 \leq i \leq m$, $\delta_i$ and its slice $s_i$ induce a $\mathbb{Z}$-grading on $B$ such that $\deg s_i = 1$ and $\deg a = 0$ for $a \in B^{\delta_i} = \{0\}$. There is an algebraic $G_m$-action on $Y$ corresponding to the $\mathbb{Z}$-grading on $B$ induced by $\delta_i$ and $s_i$. We call the action the $T_i$-action where $T_i = G_m$. The $T_i$-action on $Y$ is given by the $k$-algebra homomorphism $\rho_i : B \to B[t, t^{-1}]$ defined by (cf. Freudenburg [6, 10.2])

$$\rho_i(b) = \sum_{j \geq 0} \frac{(t - 1)^j}{j!} \delta_i^j(b)s_i^j \quad \text{for} \quad b \in B.$$

The $T_i$-actions commute with each other and the $m$-dimensional torus $T' = T_1 \times \cdots \times T_m$ acts on $Y$. The subalgebra $B^{T_i}$ is equal to $B^{\delta_i} \cong B/(s_i)$ and $B^{T'} = A$. Note that the $G$-action commutes with every $T_i$-action and $G \times T'$ acts on $Y$.

For the remainder of this section, we suppose that $k$ is algebraically closed. Let $p : Y \to Y//G := \text{Spec } B^G$ be the algebraic quotient, which is defined by the inclusion $B^G \hookrightarrow B$. Then $p$ induces an embedding of the fixed point locus $Y^G$ into $Y//G$. The $G$-action on $Y$ is called fix-pointed if $p|_{Y^G}$ is an isomorphism, i.e., every closed orbit of $Y$ is a fixed point. It is known by Bass and Haboush [2] that $Y \cong Y^G \times V$ for a $G$-representation $V$ if $Y$ is smooth, the $G$-action on $Y$ is fix-pointed, and every vector bundle over $Y$ is trivial. Hence if $Y^G \cong Y//G$ is the affine space, then $Y$ is isomorphic to the affine space and the $G$-action on $Y$ is linearizable.
Lemma 2.2. Under the notation and assumption in Lemma 2.1, suppose that \( Y = \text{Spec} \, B \) is smooth and the \( G \)-action on \( Y \) is fix-pointed. Suppose, further, that every vector bundle over \( Y \) is trivial. Then \( A \) is a polynomial ring over \( A^G \) with semi-invariant coordinates. Hence if \( A^G \) is a polynomial ring, then \( A \) is a polynomial ring over \( k \) with semi-invariant coordinates, i.e., \( X = \text{Spec} \, A \) is isomorphic to a \( G \)-representation.

Proof. By Lemma 2.1, it follows that \( Y \cong X \times V \) for a \( G \)-representation \( V \). Hence \( X \) is a smooth \( G \)-subvariety of \( Y \). Note that the \( G \)-action on \( X \) is fix-pointed as well. Further, every vector bundle over \( X \) is trivial since every vector bundle over \( Y \) is trivial. Hence it follows from [2] that \( X \cong X//G \times V' \) for a \( G \)-representation \( V' \).

If the \( G \)-action on \( Y \) is fix-pointed, then the \( G \)-action on \( X_i = \text{Spec} \, A_i \) is fix-pointed as well where \( A_i = B^{(i)} \).

If \( G = G_m \), then \( B \) is \( \mathbb{Z} \)-graded and \( B = \oplus_{i \in \mathbb{Z}} B_i \). Suppose that \( B \) is positively graded, i.e., \( B_i = 0 \) for every \( i < 0 \). Then the \( G \)-action on \( Y \) is fix-pointed. Hence \( A \) is a polynomial ring with homogeneous coordinates if \( A^G \) is a polynomial ring.

Let \( B \) be an affine domain with a \( G \)-action and let \( R = B[x_1, \ldots, x_n] \) be a polynomial ring over \( B \) with an indeterminant \( x_j \) is semi-invariant of weight \( \beta_j \). We assume that the \( G \)-action on \( R \) restricts to the action on \( B \). Let \( \delta \) be a semi-invariant locally nilpotent \( B \)-derivation on \( R \) of weight \( -\beta_1 \) and let \( f_j = \delta(x_j) \in B \) for \( 1 \leq j \leq n \). Then each \( f_j \) is semi-invariant of weight \( \beta_j - \beta_1 \). Suppose that \( \delta \) has a slice. Then the sequence \( (f_1, \ldots, f_n) \) is a unimodular row. Hence the sequence \( (f_1, \ldots, f_n) \) is considered as a surjective \( G \)-vector bundle map \( \Psi \) from the trivial \( G \)-bundle \( Y \times (L \oplus F) \) over \( Y = \text{Spec} \, B \) onto the trivial \( G \)-bundle \( Y \times L \) where \( L \) is the trivial \( G \)-representation of dimension one and \( F \) is a \( G \)-representation of weight \( \beta_2 - \beta_1, \ldots, \beta_n - \beta_1 \). In fact, \( \Psi \) is given by

\[
\Psi(y, c_1, \ldots, c_n) = (y, f_1(y)c_1 + \cdots + f_n(y)c_n)
\]

for \( y \in Y \) and \((c_1, \ldots, c_n) \in L \oplus F \). The \( G \)-action on \( B \) is given by \((g \cdot f)(y) = f(g^{-1}y) \) for \( g \in G, \ f \in B, \) and \( y \in Y \). Suppose, further, that any \( G \)-vector bundle over \( Y \) is trivial. Then the \( G \)-vector bundle obtained by the kernel of \( \Psi \) can be trivialized. Hence there exists a \( G \)-vector bundle automorphism of \( Y \times (L \oplus F) \), which is represented by an invertible matrix \( C \) of entries in semi-invariants of \( B \) such that \((f_1, \ldots, f_n) = (1, 0, \ldots, 0)C \).

Lemma 2.3. (cf. [9, Lemma 2.2]) Let \( \delta \) be a semi-invariant locally nilpotent \( B \)-derivation on \( R = B[x_1, \ldots, x_n] \) of weight \( -\beta_1 \) having a
slice and let $\delta(x_j) \in B$ for every $j$. Suppose that any $G$-vector bundle over $Y$ is trivial. Then $R^\delta$ is a polynomial ring over $B$. Furthermore, there exists a system of coordinates $\xi_2, \ldots, \xi_n$ of $R^\delta$ over $B$ such that each $\xi_i$ is semi-invariant of weight $\beta_i$ and linear in $x_1, \ldots, x_n$ over $B$.

**Proof.** Let $f_j = \delta(x_j)$ for $1 \leq j \leq n$. Then there is an invertible matrix $C$ of entries in semi-invariants of $B$ such that $(f_1, \ldots, f_n) = (1,0,\ldots,0)C$. Let $D = (d_{ij})_{1 \leq i,j \leq n}$ be the inverse of $C$. Then $d_{ij} \in B$ is semi-invariant of weight $\beta_j - \beta_i$ since $C$ represents a $G$-vector bundle automorphism of $Y \times (L \oplus F)$. Let

$$s = d_{11}x_1 + d_{21}x_2 + \cdots + d_{n1}x_n$$

$$\xi_i = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{ni}x_n \quad \text{for} \quad 2 \leq i \leq n.$$

Then $R = B[s, \xi_2, \ldots, \xi_n]$ and $s$ is of weight $\beta_1$ and $\xi_i$ of weight $\beta_i$. Since $(f_1, \ldots, f_n)D = (1,0,\ldots,0)$, it follows that $s$ is a slice of $\delta$ and $\delta(\xi_i) = 0$ for $2 \leq i \leq n$. Hence $R^\delta = B[\xi_2, \ldots, \xi_n]$ and the assertion follows.

The following is easily verified.

**Lemma 2.4.** (cf. [9, Lemma 2.3]) Let $\delta$ be a semi-invariant locally nilpotent $B$-derivation on $R = B[x_1, \ldots, x_n]$. Suppose that $\delta(x_1)$ belongs to the group $B^*$ of invertible elements of $B$. Then $R^\delta = B[x_2, \ldots, x_n]$. Hence $R^\delta$ is a polynomial ring over $B$ with semi-invariant coordinates.

### 3. Equivariant Embeddings

In this section, we observe $G$-embeddings and associate a sequence to a given $G$-embedding.

Let $V$ and $\tilde{V}$ be $G$-representations and let $\varphi : V \to \tilde{V}$ be a $G$-embedding. Let $S = k[v_1, \ldots, v_m]$ and $R = k[x_1, \ldots, x_n]$ be the coordinate rings of $V$ and $\tilde{V}$ respectively, where $v_i$ ($1 \leq i \leq m$) is semi-invariant of weight $\alpha_i$ and $X_j$ ($1 \leq j \leq n$) of weight $\beta_j$. As observed in section 1, it follows that $\tilde{V} = V \oplus V'$ for a $G$-representation $V'$. Hence we may and assume $\beta_i = \alpha_i$ for $1 \leq i \leq m$.

**Proposition 3.1.** Suppose that there exists a subgroup $G'$ of $G$ such that $\tilde{V}^{G'} = V$. Then $\varphi : V \to \tilde{V}$ is $G$-rectifiable.

**Proof.** Note that $V^{G'} = V$ since $V = \tilde{V}^{G'}$ is $G'$-invariant. Hence the $G$-embedding $\varphi$ induces an embedding $V^{G'} \to \tilde{V}^{G'}$, which is an isomorphism. Thus $\varphi$ is $G$-rectifiable. \qed
Let $\Phi : R \to S$ be the $G$-equivariant surjection associated with the $G$-embedding $\varphi : V \to \bar{V}$. Since $\Phi$ is $G$-equivariant, $\Phi$ induces a surjection $\Phi_\chi : R_\chi \to S_\chi$ for every character $\chi$ of $G$.

**Proposition 3.2.**

(1) Suppose that for $1 \leq i \leq m$, $\alpha_i$ is non-trivial and the $R^G$-module $R_{\alpha_i}$ is generated by one element. Then $\varphi$ is $G$-rectifiable.

(2) Suppose that $S_{\beta_j} = 0$ for $m+1 \leq j \leq n$. Then $\varphi$ is $G$-rectifiable.

**Proof.**

(1) It suffices to show that $\Phi(X_i) = c_i v_i$ for $1 \leq i \leq m$ where $c_i \in k^*$. In fact, if $\Phi(X_i) = c_i v_i$ with $c_i \in k^*$ for $1 \leq i \leq m$, then we have a system of semi-invariant coordinates $X_1, \ldots, X_m, X'_m, \ldots, X'_n$ of $R$ satisfying $\Phi(X'_i) = 0$ for $1 \leq j \leq n - m$ where $X'_j = X_j - f_j X_1 - \cdots - f_m X_m$ for $f_j = \Phi(X_j)$. It follows from the assumption that for $1 \leq i \leq m$, $R_{\alpha_i}$ is generated by $X_i$ over $R^G$ since $X_i$ is irreducible. Since $\Phi_{\alpha_i} : R_{\alpha_i} \to S_{\alpha_i}$ is surjective, there exists $f_i \in R^G$ such that $\Phi_{\alpha_i}(f_i X_i) = \Phi_{\alpha_i}(f_i) \Phi_{\alpha_i}(X_i) = v_i$. It follows that $\Phi(f_i) \in S^* = k^*$ since $v_i$ is irreducible and semi-invariant of non-trivial weight. Hence $\Phi(X_i) = c_i v_i$ for $c_i \in k^*$.

(2) Since $\Phi_{\beta_j} : R_{\beta_j} \to S_{\beta_j}$ is trivial, $\Phi(X_j) = 0$ for $m + 1 \leq j \leq n$. Hence the assertion follows.

We show that every $G$-embedding $V \to \bar{V}$ is $G$-rectifiable when composed by a suitable $G$-embedding $V \to V \oplus \bar{V}$.

**Proposition 3.3.** There exists a $G$-embedding $\psi : \bar{V} \to V \oplus \bar{V}$ such that $\tilde{\varphi} = \psi \circ \varphi$ is $G$-rectifiable.

**Proof.**

Since $\Phi : R \to S$ is a $G$-equivariant surjection, there exists an $H_i \in R$ of weight $\alpha_i$ such that $\Phi(H_i) = v_i$ for $1 \leq i \leq m$. Let $I$ be the kernel of $\Phi$. Then $I$ is $G$-stable and $R = k[H_1, \ldots, H_m] \oplus I$ as a $G$-module. Hence each $X_j$ is written as $X_j = \xi_j + \eta_j$ where $\xi_j \in k[H_1, \ldots, H_m]$ and $\eta_j \in I$ are of weight $\beta_j$. Thus the semi-invariant $m + n$ elements $H_1, \ldots, H_{m+n}$ generate $R$ over $k$, where $H_{m+j} := \eta_j$ for $1 \leq j \leq n$. Note that $I = (H_{m+1}, \ldots, H_{m+n})$. Let $\bar{R} = k[\bar{X}_1, \ldots, \bar{X}_{m+n}]$ be a polynomial ring with a linear $G$-action such that the weight of $\bar{X}_j$ is $\alpha_j$ for $1 \leq j \leq m$ and $\beta_{j-m}$ for $m+1 \leq j \leq m + n$. Then $\text{Spec} \ \bar{R}$ is isomorphic to the $G$-representation $V \oplus \bar{V}$. Let $\Psi : \bar{R} \to R$ be the $G$-equivariant surjection defined by $\Psi(\bar{X}_j) = H_j$ for $1 \leq j \leq m + n$. Then $\Psi$ defines a $G$-embedding $\psi : \bar{V} \to V \oplus \bar{V}$ and the kernel of the $G$-equivariant surjection $\bar{\Phi} := \Phi \circ \Psi$ is the $G$-stable ideal generated by $\bar{X}_{m+1}, \ldots, \bar{X}_{m+n}$. Hence the $G$-embedding $\tilde{\varphi} = \psi \circ \varphi : V \to V \oplus \bar{V}$ is $G$-rectifiable and the assertion follows.
We shall associate a sequence $\Delta_\varphi$ to the $G$-embedding $\varphi : V \to \tilde{V}$. Let $B$ be a polynomial ring $k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]$ with a linear $(G \times T)$-action where $T = G_m$ as stated in section 1. The weight of $u$, $v_i$ ($1 \leq i \leq m$), $x_j$ ($1 \leq j \leq n$) is $(0, -1)$, $(\alpha_i, 0)$, $(\beta_j, d)$, respectively, where $d$ is a positive integer. Let $f_j$ be the image of $X_j$ by $\Phi : R \to S$ for $1 \leq j \leq n$. We define a derivation $\delta_i$ on $B$ by

$$\delta_i = u^d \partial_v + (\partial_v f_1) \partial_{x_1} + \cdots + (\partial_v f_n) \partial_{x_n}$$

for $1 \leq i \leq m$. Then $\delta_i$ is a semi-invariant locally nilpotent derivation of weight $(-\alpha_i, -d)$ and $\delta_i \delta_l = \delta_l \delta_i$ for any $i$ and $l$. Let $\Delta_\varphi = (\delta_1, \ldots, \delta_m)$.

**Proposition 3.4.** ([9, Proposition 3.2]) The sequence $\Delta_\varphi = (\delta_1, \ldots, \delta_m)$ has a semi-invariant slice system $(s_1, \ldots, s_m)$.

**Proof.** We show that each $\delta_i$ has a slice $s_i$ of weight $(\alpha_i, d)$ such that $\delta_i(s_i) = 0$ for $i \neq l$. Since $\Phi$ is a surjection, there exists a polynomial $F_i \in k[X_1, \ldots, X_n]$ such that

$$F_i(f_1, \ldots, f_n) = v_i$$

for $1 \leq i \leq m$. Let $s_i \in B$ be an element satisfying

$$F_i(f_1 - u^d x_1, \ldots, f_n - u^d x_n) = v_i - u^d s_i.$$ 

Since $\delta_i(f_j - u^d x_j) = 0$ for $1 \leq l \leq m$ and $1 \leq j \leq n$, it follows that $\delta_i(v_i - u^d s_i) = 0$, i.e., $\delta_i(s_i) = 1$ and $\delta_i(s_i) = 0$ for $i \neq l$. We can take $s_i$ to be semi-invariant of weight $(\alpha_i, d)$, and the assertion follows. \qed

We call $\Delta_\varphi$ the sequence associated to a $G$-embedding $\varphi$.

**4. Sequences and $G$-embeddings**

In this section, we show that there is a bijective correspondence between the $G$-equivalence classes of $G$-embeddings and the $G$-equivalence classes of sequences having slice systems.

Let

$$B = k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]$$

where the weight of $u$, $v_i$, $x_j$ is $(0, -1)$, $(\alpha_i, 0)$, $(\beta_j, d)$, respectively, as in the previous section. We assume that $\beta_i = \alpha_i$ for $1 \leq i \leq m$. Note that

$$B^T = k[v_1, \ldots, v_m, u^d x_1, \ldots, u^d x_n].$$

Let $Y = \text{Spec } B$. As a $(G \times T)$-representation,

$$Y = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d}.$$ 

Here $L = \text{Spec } k[u]$, $V_0 = \text{Spec } S$, $V_{-d} = \text{Spec } k[x_1, \ldots, x_m]$, and $V'_{-d} = \text{Spec } k[x_{m+1}, \ldots, x_n]$ where $S = k[v_1, \ldots, v_m]$. As a $G$-representation,
$V_0 = V = V_d$. We shall observe the kernel of a sequence. Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence, i.e., a sequence of mutually commuting locally nilpotent derivations on $B$ of a form

$$\delta_i = u^d\partial v_i + f_{i1}\partial x_1 + \cdots + f_{in}\partial x_n$$

(1)

where $f_{ij} \in B$ is semi-invariant of weight $(\beta_j - \alpha_i, 0)$ for $1 \leq j \leq n$. Each $\delta_i$ is semi-invariant of weight $(-\alpha_i, -d)$. Let

$$A = \bigcap_{i=1}^m B^{\delta_i}.$$ 

The sequence $\Delta = (\delta_1, \ldots, \delta_m)$ uniquely extends to a sequence of mutually commuting semi-invariant locally nilpotent derivations on the localization $B_u$ at $u$. Note that the $(G \times T)$-action on $B$ extends to the action on $B_u$. The kernel $(B_u)^{\delta_i}$ is the localization of $B^{\delta_i}$ at $u$, and $\cap_{i=1}^m (B_u)^{\delta_i} = A_u$. The sequence $\Delta = (\delta_1, \ldots, \delta_m)$ of the derivations on $B_u$ has a semi-invariant slice system $(v_1/u^d, \ldots, v_m/u^d)$. Hence it follows from Lemma 2.1 that $B_u = A_u[v_1, \ldots, v_m]$ and

$$A_u = k[u, u^{-1}, \phi(x_1), \ldots, \phi(x_n)]$$

where $\phi = \phi_1 \circ \cdots \circ \phi_1$ and $\phi_i$ is the Dixmier map defined by

$$\phi_i(b) = b + \sum_{l \geq 1} \frac{(-1)^l}{l!} \delta_i^l(b) \left( \frac{v_i}{u^d} \right)^l \quad \text{for} \quad b \in B_u.$$ 

Note that each $\phi_i$ preserves the weight since $\delta_i$ is semi-invariant of weight $(-\alpha_i, -d)$ and $v_i/u^d$ is of weight $(\alpha_i, d)$. The following is easily proved (cf. [9]).

**Lemma 4.1.** (1) For $1 \leq j \leq n$, $u^d\phi(x_j)$ is uniquely written as

$$u^d\phi(x_j) = u^d\theta_j + h_j$$

where $\theta_j \in B$ is an element of weight $(\beta_j, d)$ satisfying $u^d\theta_j \in B^T \cap u^dB$ and $h_j \in S$ is of weight $(\beta_j, 0)$.

(2)

$$A^T = k[u^d\phi(x_1), \ldots, u^d\phi(x_n)]$$

$$= k[u^d\theta_1 + h_1, \ldots, u^d\theta_n + h_n].$$

Hence $A^T$ is a polynomial ring in semi-invariant $n$ variables.

Note that $\text{Spec } A^T$ is $G$-equivariantly isomorphic to $\tilde{V} = \text{Spec } R$. Let $\tilde{B} = B/(u) \cong k[v_1, \ldots, v_m, x_1, \ldots, x_n]$. Then $\tilde{B}$ inherits the $(G \times T)$-action and the surjection $q : B \to \tilde{B}$ preserves the weight. The derivation $\delta_i$ induces a semi-invariant locally nilpotent derivation
$\tilde{d}_i$ on $\tilde{B}$. Let $\tilde{A} = \cap_{i=1}^m \tilde{B}^{\tilde{d}_i}$. Since $q \circ \tilde{d}_i = \tilde{d}_i \circ q$ for every $i$, $q$ induces an algebra homomorphism $q_0 : A \to \tilde{A}$ which preserves the weight.

**Lemma 4.2.** ([9, Lemma 3.1]) The following are equivalent.

1. The sequence $\Delta$ has a slice system.
2. The sequence $\Delta$ has a semi-invariant slice system $(s_1, \ldots, s_m)$ such that $s_i$ is of weight $(\alpha_i, d)$.
3. The homomorphism $q_0 : A \to \tilde{A}$ is surjective.

In the following, a semi-invariant slice system $(s_1, \ldots, s_m)$ of $\Delta$ implies that $(s_1, \ldots, s_m)$ is a slice system of $\Delta$ such that $s_i$ is semi-invariant of weight $(\alpha_i, d)$ for $1 \leq i \leq m$. Suppose that $\Delta = (\delta_1, \ldots, \delta_m)$ has a semi-invariant slice system $(s_1, \ldots, s_m)$. Then $v_i - u^d s_i \in A^T$ for every $i$. By Lemma 4.1 (2), there exists a polynomial $F_i \in k[X_1, \ldots, X_n]$ for $1 \leq i \leq m$ such that

$$v_i - u^d s_i = F_i(u^d \theta_1 + h_1, \ldots, u^d \theta_n + h_n).$$

Substituting $u = 0$ to the above equation, we obtain

$$v_i = F_i(h_1, \ldots, h_n). \quad (2)$$

We consider $R = k[X_1, \ldots, X_n]$ as a polynomial ring with a linear $(G \times T)$-action such that $X_j$ is of weight $(\beta_j, 0)$ for $1 \leq j \leq n$. As a $G$-representation, Spec $R = \tilde{V}$. We define a $G$-equivariant algebra homomorphism $\Phi_\Delta : R \to S$ by

$$\Phi_\Delta(X_j) = h_j \quad \text{for } 1 \leq j \leq n.$$ 

Since $\Phi_\Delta$ is a surjection by (2), it defines a $G$-embedding $\varphi_\Delta : V \to \tilde{V}$. In particular, we have $n \geq m$. We call $\varphi_\Delta$ (resp. $\Phi_\Delta$) the $G$-embedding (resp. $G$-equivariant surjection) associated to $\Delta$.

**Proposition 4.3.** Let $\varphi : V \to \tilde{V}$ be a $G$-embedding and let $\Delta_\varphi = (\delta_1, \ldots, \delta_m)$ be the sequence associated to $\varphi$. Then the $G$-embedding $\varphi_\Delta$ associated to $\Delta_\varphi$ is $G$-equivalent to $\varphi$.

**Proof.** Let $\Phi : R \to S$ be the $G$-equivariant surjection associated with $\varphi$. Recall that $\delta_i$ of $\Delta_\varphi$ is defined by

$$\delta_i = u^d \partial_{v_i} + (\partial_{v_i} f_1) \partial_{x_1} + \cdots + (\partial_{v_i} f_n) \partial_{x_n}$$

where $f_j = \Phi(X_j)$ for $1 \leq j \leq n$. The associated surjection $\Phi_{\Delta_\varphi} : R \to S$ is defined by $\Phi_{\Delta_\varphi}(X_j) = h_j$ where $h_j \in S$ is a polynomial satisfying $u^d \phi(x_j) = u^d \theta_i + h_j$ for $u^d \theta_i \in B^T$. It is shown in [9, Proposition 3.2] that $h_j = -f_j + c_j$ for some $c_j \in k$. Since both of the weight of $h_j$ and of $f_j$ are equal to $(\beta_j, 0)$, it follows that $c_j = 0$ if $\beta_j$ is non-trivial.
Hence by a $G$-equivariant affine automorphism $\gamma$ of $R$, it follows that $\Phi_{\Delta, \varphi} = \Phi \circ \gamma$, i.e., $\varphi_{\Delta, \varphi}$ is $G$-equivalent to $\varphi$. \hfill $\square$

Suppose that $\Delta = (\delta_1, \ldots, \delta_m)$ has a slice system. Then the associated $G$-embedding $\varphi_{\Delta}$, equivalently, the $G$-equivariant surjection $\Phi_{\Delta} : R \to S$ can be defined. Since $\Phi_{\Delta}$ is a surjection, there exists an $H_i \in R$ of weight $\alpha_i$ such that $\Phi_{\Delta}(H_i) = v_i$ for $1 \leq i \leq m$. Let $I$ be the kernel of $\Phi_{\Delta}$. Then $I$ is $G$-stable and $R = k[H_1, \ldots, H_m] \oplus I$ as a $G$-module. There is a set of semi-invariant generators $H_{m+1}, \ldots, H_{m+n}$ of $I$ such that $H_1, \ldots, H_{m+n}$ generate $R$ over $k$ (cf. the proof of Proposition 3.3). We can choose appropriate generators among $H_{m+1}, \ldots, H_{m+n}$ and change their subscripts if necessary so that $I = (H_{m+1}, \ldots, H_r)$ and $H_1, \ldots, H_m, H_{m+1}, \ldots, H_r$ generate $R$. Note that $r \geq n$. The weight of $H_i$ is $\gamma_i$ where $\gamma_i = \alpha_i$ if $1 \leq i \leq m$ and $\gamma_i = \beta_j$ for some $j$ if $m + 1 \leq i \leq r$. We have

$$\Phi_{\Delta}(H_i) = H_i[h_1, \ldots, h_n] = \begin{cases} v_i & \text{for } 1 \leq i \leq m \\ 0 & \text{for } m + 1 \leq i \leq r. \end{cases}$$

Note that if $\varphi_{\Delta}$ is $G$-rectifiable, then we can take $r = n$ and $H_1, \ldots, H_n$ is a system of semi-invariant coordinates of $R$.

Let

$$\sigma : R \to A^T = k[u^d\theta_1 + h_1, \ldots, u^d\theta_n + h_n]$$

be the $G$-equivariant isomorphism defined by $X_i \mapsto u^d\theta_i + h_i$. We define $w_i$ ($1 \leq i \leq m$) and $y_j$ ($m + 1 \leq j \leq r$) by

$$\sigma(H_i) = H_i(u^d\theta_1 + h_1, \ldots, u^d\theta_n + h_n) = \begin{cases} v_i - u^d w_i & \text{for } 1 \leq i \leq m \\ u^d y_i & \text{for } m + 1 \leq i \leq r. \end{cases}$$

Since $\sigma(H_i) \in A^T$, it follows that for every $l$

$$\delta_l(v_i - u^d w_i) = \delta_l(u^d y_j) = 0.$$

Hence $(w_1, \ldots, w_m)$ is a semi-invariant slice system of $\Delta$, and $y_j$ ($m + 1 \leq j \leq r$) is an element of $A$ of weight $(\gamma_j, d)$. Since $H_1, \ldots, H_r$ generate $R$, it follows from $A^T = \sigma(R)$ that

$$A^T = k[\tilde{v}_1, \ldots, \tilde{v}_m, u^d y_{m+1}, \ldots, u^d y_r]$$

where $\tilde{v}_i := v_i - u^d w_i$ for $1 \leq i \leq m$.

Lemma 4.4. (cf. [9, Lemmas 3.3 and 3.5]) Suppose that $\Delta = (\delta_1, \ldots, \delta_m)$ has a slice system.

(1) $A = k[u, \tilde{v}_1, \ldots, \tilde{v}_m, y_{m+1}, \ldots, y_r]$. 
As a consequence, if the associated $G$-embedding $\varphi_\Delta$ is $G$-rectifiable, then $A$ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action.

(2) \[ A = k[u, u^d \theta_1 + h_1, \ldots, u^d \theta_n + h_n, u^{-d} I(u^d \theta + h)] \]
where $I(u^d \theta + h) = \sigma(I)$ is the ideal of $A^T$.

(3) There is a $k[u]$-algebra isomorphism
\[ \tilde{\sigma} : R[u, u^{-d} I] \xrightarrow{\sim} A \]
which is $(G \times T)$-equivariant and restricts to the isomorphism $\sigma : R \xrightarrow{\sim} A^T$. Here, we consider $R[u, u^{-1} I]$ as the $(G \times T)$-subalgebra of $R[u, u^{-1}]$.

When the $G$-action on $Y$ is trivial, we obtain by Lemma 4.4 that the kernel of a sequence $\Delta$ with a slice system is a polynomial ring with semi-invariant coordinates with respect to the $T$-action if the associated embedding $\varphi_\Delta$ is rectifiable. Forgetting the $G$-action, it is well-known that $\varphi_\Delta : V \to \tilde{V}$ is rectifiable in the cases (1) $m = 1$ and $n = 2$, (2) $m = n$, (3) $n \geq 2m + 2$. Hence in these three cases, $A$ is a polynomial ring and $u$ is a coordinate variable of $A$. In the case (1), it follows that $A = k[u, \tilde{v}, y]$ where $\tilde{v}$ (resp. $y$) is of degree 0 (resp. $d$) with respect to the $T$-action.

Corollary 4.5. Suppose that $k = \mathbb{C}$ and $G = \mathbb{C}^*$. If $m = 1$ and $n = 2$, and a semi-invariant locally nilpotent derivation $\delta$ on $B = \mathbb{C}[u,v,x_1,x_2]$ of a form (1) has a slice, then $A$ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action.

Proof. Let $X = \text{Spec } A$. It follows from Lemma 4.1 (2) that $X/\!/T = \text{Spec } A^T \cong \mathbb{A}^2$. Hence $\dim X/\!\!/ (G \times T) \leq 2$. If $\dim X/\!\!/ (G \times T) = 2$, then the $G$-action on $X \cong \mathbb{A}^3$ is trivial, and we are done. If $X/\!\!/ (G \times T)$ is one-dimensional, the $(G \times T)$-action on $X$ is linearizable by [8]. If $X/\!\!/ (G \times T)$ is a point, then the $(G \times T)$-action on $X$ is linearizable by [2].

We show that there is a bijective correspondence between the $G$-equivalence classes of $G$-embeddings and the $G$-equivalence classes of sequences with slice systems.

Lemma 4.6. (cf. [9, Lemma 3.6]) Let $\Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)})$ and $\Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)})$ be two sequences having slice systems. Let $\varphi_1$ and $\varphi_2$ be the $G$-embeddings associated to $\Delta_1$ and $\Delta_2$, respectively. Then $\varphi_1$ and $\varphi_2$ are $G$-equivalent if and only if $\Delta_1$ and $\Delta_2$ are $G$-equivalent.
Proof. Suppose that $\Delta_1$ and $\Delta_2$ are $G$-equivalent. Then there is a $(G \times T)$-equivariant $S$-automorphism $\psi$ of $B$ such that $\delta_i^{(2)} \circ \psi = \psi \circ \sigma_i^{(1)}$ for every $i$. Let $A_{(1)} = \cap_{i=1}^m B_{i}^{(1)}$ and $A_{(2)} = \cap_{i=1}^m B_{i}^{(2)}$. Then $\psi$ induces a $G$-equivariant isomorphism $\psi|_{A_{(1)}} : A_{(1)}^T \xrightarrow{\sim} A_{(2)}^T$. Let $u^d\theta_j^{(1)} + h_j^{(1)} (1 \leq j \leq n)$ be the element of $A_{(1)}^T$ defined as in Lemma 4.1 (1) with respect to $\Delta_i$ for $l = 1, 2$. Then the $G$-equivariant isomorphism $\sigma_l : R \xrightarrow{\sim} A_{(1)}^T$ is defined by $\sigma_l(x_j) = u^d\theta_j^{(1)} + h_j^{(1)}$ for $1 \leq j \leq n$. The isomorphism $\psi|_{A_{(1)}}$ induces a $G$-equivariant automorphism $\gamma : R \to R$ such that $\gamma \circ \sigma_l^{-1} = \sigma_2^{-1} \circ \psi|_{A_{(2)}^T}$. Let $\Phi_l : R \to S$ be the $G$-equivariant surjection associated to $\Delta_i$. Then noting that $\psi(u) = cu$ for $c \in k^*$, one obtains that for every $j$,

$$(\Phi_2 \circ \gamma \circ \sigma_l^{-1})(u^d\theta_j^{(1)} + h_j^{(1)}) = h_j^{(1)} = (\Phi_1 \circ \sigma_l^{-1})(u^d\theta_j^{(1)} + h_j^{(1)})$$

(cf. the proof of [9, Lemma 3.6]). Hence it follows that $\Phi_1 = \Phi_2 \circ \gamma$, and $\varphi_1$ and $\varphi_2$ are $G$-equivalent.

Conversely, suppose that $\varphi_1$ and $\varphi_2$ are $G$-equivalent, i.e., $\Phi_1 = \Phi_2 \circ \gamma$ for a $G$-equivariant automorphism $\gamma$ of $R$. Then it follows that $I_2 = \gamma(I_1)$ where $I_1 \subset R$ is the kernel of $\Phi_1$. Hence $\gamma$ extends to a $k[u]$-isomorphism $\tilde{\gamma} : R[u, u^{-d}I_1] \xrightarrow{\sim} R[u, u^{-d}I_2]$ which is $(G \times T)$-equivariant. By Lemma 4.4 (3), $\tilde{\gamma}$ induces a $k[u]$-isomorphism $\psi : A_{(1)} \xrightarrow{\sim} A_{(2)}$ which is $(G \times T)$-equivariant and satisfies $\sigma_2 \circ \gamma = \psi \circ \sigma_1$. Let $H_i (1 \leq i \leq m)$ be the element of $R$ of weight $\alpha_i$ such that $\Phi_1(H_i) = v_i$, and let $w_i^{(1)} \in B (1 \leq i \leq m)$ be an element of degree $(\alpha_i, d)$ defined by $v_i - u^d w_i^{(1)} = \sigma_1(H_i)$. Then since $(w_1^{(1)}, \ldots, w_m^{(1)})$ is the semi-invariant slice system of $\Delta_1$, it follows that $B = A_{(1)}[w_1^{(1)}, \ldots, w_m^{(1)}]$. We extend the $(G \times T)$-equivariant $k[u]$-isomorphism $\psi : A_{(1)} \to A_{(2)}$ to a $(G \times T)$-equivariant algebra homomorphism $\psi : B \to B$ by defining $\psi(w_i^{(1)})$ for $1 \leq i \leq m$ by

$$u^d \psi(w_i^{(1)}) = v_i - (\psi \circ \sigma_1)(H_i).$$

Note that $v_i - (\psi \circ \sigma_1)(H_i) \in u^d B$. In fact, since

$$(\psi \circ \sigma_1)(H_i) = (\sigma_2 \circ \gamma)(H_i) = (\gamma(H_i))(u^d\theta_1^{(2)} + h_1^{(2)}, \ldots, u^d\theta_n^{(2)} + h_n^{(2)})$$

and

$$(\gamma(H_i))(h_1^{(2)}, \ldots, h_n^{(2)}) = \Phi_2(\gamma(H_i)) = \Phi_1(H_i) = v_i,$$

it follows that $v_i - (\psi \circ \sigma_1)(H_i) \in u^d B$ and $\psi(w_i^{(1)}) \in B$ is defined. Let $w_i^{(2)} = \psi(w_i^{(1)})$ for $1 \leq i \leq m$. Then by the definition of $\psi(w_i^{(1)})$, it is easily checked that $(w_1^{(2)}, \ldots, w_m^{(2)})$ is a semi-invariant slice system of
Δ₂. Hence \( B = A_{(2)}[w^{(2)}_1, \ldots, w^{(2)}_m] \). It follows that \( \psi \) is a \((G \times T)\)-equivariant automorphism of \( B \) which satisfies \( \delta^{(2)}_i \circ \psi = \psi \circ \delta^{(1)}_i \) for every \( i \). Furthermore, \( \psi \) is a \( S \)-automorphism of \( B \) since it follows from \( v_i = u^d w^{(1)}_i + \sigma_1(H_i) \) for \( 1 \leq i \leq m \) that
\[
\psi(v_i) = u^d \psi(w^{(1)}_i) + (\psi \circ \sigma_1)(H_i) = v_i - (\psi \circ \sigma_1)(H_i) + (\psi \circ \sigma_1)(H_i) = v_i.
\]

By Proposition 4.3 and Lemma 4.6, there is a bijective correspondence between the \( G \)-equivalence classes of \( G \)-embeddings \( V \to \tilde{V} \) and the \( G \)-equivalence classes of sequences \( \Delta = (\delta_1, \ldots, \delta_m) \) with slice systems.

For two sequences \( \Delta_1 = (\delta^{(1)}_1, \ldots, \delta^{(1)}_m) \) and \( \Delta_2 = (\delta^{(2)}_1, \ldots, \delta^{(2)}_m) \), we define that \( \Delta_1 \) and \( \Delta_2 \) are weakly \( G \)-equivalent iff there is a \((G \times T)\)-equivariant automorphism \( \psi \) of \( B \), not necessarily an \( S \)-automorphism, such that \( \delta^{(2)}_i \circ \psi = \psi \circ \delta^{(1)}_i \) for every \( i \). We define also that two \( G \)-equivariant embeddings \( \varphi_1 \) and \( \varphi_2 \) of \( V \) into \( \tilde{V} \) are weakly \( G \)-equivalent iff there is a \( G \)-equivariant automorphism \( \gamma \) of \( R \) such that \( I_2 = \gamma(I_1) \) where \( I_l \) is the kernel of the surjection \( \Phi_l : R \to S \) associated with \( \varphi_l \) for \( l = 1, 2 \). Then there is a bijective correspondence between the weakly \( G \)-equivalence classes of \( G \)-embeddings \( V \to \tilde{V} \) and the weakly \( G \)-equivalence classes of sequences \( \Delta = (\delta_1, \ldots, \delta_m) \) with slice systems. In fact, Lemma 4.6 holds true when replacing \( "G\text{-equivalent}" \) by \( "\text{weakly } G\text{-equivalent}" \). The difference between \( "G\text{-equivalent}" \) and \( "\text{weakly } G\text{-equivalent}" \) is whether one admits the exchange of a system of semi-invariant coordinates of \( V = \text{Spec } S \) or not. Note that a \( G \)-embedding \( \varphi : V \to \tilde{V} \) is \( G \)-rectifiable iff \( \varphi \) is weakly \( G \)-equivalent to the standard \( G \)-embedding \( V \to V \oplus V' \to \tilde{V} \).

**Lemma 4.7.** Let \( \Delta_l = (\delta^{(l)}_1, \ldots, \delta^{(l)}_m) \) and \( \varphi_l \) be the same as in Lemma 4.6 for \( l = 1, 2 \). Let \( A_{(l)} = \cap_{i=1}^m B^{(l)}_{A_i} \) for \( l = 1, 2 \). Then the following are equivalent.

1. \( \varphi_1 \) and \( \varphi_2 \) are weakly \( G \)-equivalent.
2. \( \Delta_1 \) and \( \Delta_2 \) are weakly \( G \)-equivalent.
3. \( A_{(1)} \) is \((G \times T)\)-equivariantly isomorphic to \( A_{(2)} \).

**Proof.** The equivalence between (1) and (2) can be proved as in the proof of [9, Lemma 3.7]. It is obvious that (3) follows from (2). We show that (3) implies (2). Suppose that (3) holds. Then there
is a \((G \times T)\)-equivariant isomorphism \(\psi : A_{(1)} \sim A_{(2)}\). The isomorphism \(\psi\) extends to a \((G \times T)\)-automorphism \(\psi\) of \(B\) such that 
\[
\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)} \quad \text{for every } i.
\]
In fact, let \((w_1^{(1)}, \ldots, w_m^{(1)})\) (resp. \((w_1^{(2)}, \ldots, w_m^{(2)})\)) be any semi-invariant slice system of \(\Delta_1\) (resp. \(\Delta_2\)). Then \(B = A_{(1)}[w_1^{(1)}, \ldots, w_m^{(1)}] = A_{(2)}[w_1^{(2)}, \ldots, w_m^{(2)}]\). Define an algebra homomorphism 
\[
\psi : B \to B
\]
by extending \(\psi : A_{(1)} \to A_{(2)}\) by 
\[
\psi(w_i^{(1)}) = w_i^{(2)}.
\]
Then \(\psi : B \to B\) is the required automorphism and \(\Delta_1\) and \(\Delta_2\) are weakly \(G\)-equivalent. 

5. The \((G \times T)\)-action on \(X\) and \(G\)-embeddings

In this section, we continue the notation in the previous section and observe the \(G\)-embedding \(\varphi_{\Delta}\) associated to a sequence on \(B\) and the \((G \times T)\)-action on \(X = \text{Spec } A\).

Let \(\Delta = (\delta_1, \ldots, \delta_m)\) be a sequence with a slice system on \(B = k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]\). Then as observed in the previous section, it follows that \(m \leq n\). Let \((w_1, \ldots, w_m)\) be a semi-invariant slice system of \(\Delta\). Then it follows that \(B = A[w_1, \ldots, w_m]\), i.e.,
\[
X \times V_{-d} \cong Y = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d}
\]  \tag{3}
where \(X = \text{Spec } A\). If \(X\) is isomorphic to a \((G \times T)\)-representation, then it follows from (3) that \(X\) is necessarily isomorphic to the \((G \times T)\)-representation
\[
W := L \oplus V_0 \oplus V'_{-d}.
\]
If the associated \(G\)-embedding \(\varphi_{\Delta}\) is \(G\)-rectifiable, then it follows from Lemma 4.4 that \(X \cong W\), namely, the equivariant cancellation holds. By Lemma 4.7, it follows that the associated \(G\)-embedding \(\varphi_{\Delta}\) is \(G\)-rectifiable if and only if \(X \cong W\).

**Theorem 5.1.** (cf. [9, Theorem 5.1]) Suppose that a sequence \(\Delta\) has a slice system. Then \(X\) is isomorphic to \(W\) if and only if the associated \(G\)-embedding \(\varphi_{\Delta}\) is \(G\)-rectifiable.

As observed in section 2, the \(m\)-dimensional torus \(T'\) acts on \(Y = \text{Spec } B\) where \(T' = T_1 \times \cdots \times T_m\) with \(T_i = G_m\) for \(1 \leq i \leq m\). The \(T_i\)-action on \(Y\) corresponds to the \(\mathbb{Z}\)-grading on \(B = A[w_1, \ldots, w_m]\) such that \(\deg w_i = 1\) and \(\deg a = 0\) for \(a \in B^{\delta_i} - \{0\}\). Since the \(T'\)-action on \(Y\) commutes with the \((G \times T)\)-action, an \((m+1)\)-dimensional torus \((T \times T')\) acts on \(Y\). Note that \(B^{T_i} = B^{\delta_i}\) and \(B^{T'} = A\). Hence \(X = Y//T'\). The algebraic quotient \(Y//T' = \text{Spec } B^{T'}\) is \((G \times T)\)-equivariantly isomorphic to the fixed-point locus \(Y^{T'} = \text{Spec } B/(w_1, \ldots, w_m)\).
Proposition 5.2. The \((G \times T \times T')\)-action on \(Y\) is linearizable if and only if \(X \cong W\).

Proof. If the \((G \times T \times T')\)-action on \(Y\) is linearizable, then \(X = \frac{Y}{W}\) is isomorphic to a \((G \times T)\)-representation, which must be \(W\). Conversely, suppose that \(X \cong W\). Then since the \((G \times T \times T')\)-variety \(Y\) is a product of a \((G \times T)\)-variety \(X\) with a trivial \(T'\)-action and a \((G \times T \times T')\)-representation, the \((G \times T \times T')\)-action on \(Y\) is linearizable.

By Theorem 5.1 and Proposition 5.2, we have the following.

Corollary 5.3. The \((G \times T \times T')\)-action on \(Y\) is linearizable if and only if \(\varphi_\Delta\) is \(G\)-rectifiable.

It is known that the \((T \times T')\)-action on \(Y\) is linearizable iff \(\varphi_\Delta\) is rectifiable (cf. [9, Lemma 4.1]).

In the following, we assume that \(k\) is algebraically closed. Note that \(W\) is isomorphic to \(L \sim V\) as a \(G\)-representation.

Proposition 5.4. Let \(\Delta\) be a sequence on \(B\) having a slice system. Suppose that the \(G\)-action on \(Y\) is fix-pointed and \(V^G = \{0\}\). Then \(X \cong L \oplus \tilde{V}\) as a \(G\)-representation.

Proof. By Lemma 2.2, it follows that \(X \cong \frac{X}{G} \cong W'\) for a \(G\)-representation \(W'\). Note that \(\frac{X}{G} \cong X^G\) since the \(G\)-action on \(X\) is fix-pointed as well. Since \(V^G = \{0\}\), it follows from (3) that \(X^G \cong L \oplus \tilde{V}^G\), hence \(\frac{X}{G}\) is an affine space. Therefore \(X\) is isomorphic to a \(G\)-representation, hence \(X \cong L \oplus \tilde{V}\).

Remark. Under the assumption in Proposition 5.4, the \(G\)-action on \(X_i = \text{Spec} A_i\) is fix-pointed where \(A_i = B^{k_i}\). Hence it follows that every \(X_i\) is isomorphic to a \(G\)-representation.

Suppose that \(G\) is an \(r\)-dimensional torus \((k^*)^r\). Then \(B\) is \(\mathbb{Z}^r\)-graded. If every component of \(\deg v_i\) is positive and every component of \(\deg x_j\) is non-negative, then the \(G\)-action on \(Y\) is fix-pointed and \(V^G = \{0\}\). By Proposition 5.4, it follows that \(X \cong L \oplus \tilde{V}\).

We obtain the following result on \(G\)-embeddings.

Theorem 5.5. Let \(G\) be a complex torus. Let \(V\) and \(\tilde{V}\) be complex \(G\)-representations and let \(\dim \tilde{V}/G \leq 1\). Then every \(G\)-embedding \(V \to \tilde{V}\) is \(G\)-rectifiable.

Proof. Let \(\varphi : V \to \tilde{V}\) be a \(G\)-embedding. By Theorem 5.1 and Proposition 4.3, it suffices to show that \(X = \text{Spec} A\) is isomorphic to
a \((G \times T)\)-representation where \(A\) is the kernel of the sequence \(\Delta_\varphi\) associated to \(\varphi\). Note that \(X\) is smooth and acyclic by (3). Recall that \(X//T = \text{Spec} A^T\) is isomorphic to the \(G\)-representation \(\hat{V}\). Hence the algebraic quotient \(X//((G \times T)) \cong \hat{V} // G\) is of dimension \(\leq 1\). By the results of [2] and [8], \(X\) is isomorphic to the \((G \times T)\)-representation. Hence the assertion follows. \(\square\)

Let \(p_L : X \to L\) be the morphism corresponding to the inclusion \(k[u] \to A\). Then \(p_L\) is \(G\)-equivariant. Since the derivation \(\delta_i\) on \(B_u\) has a slice \(v_i/u^d_i\), it follows from Lemma 2.4 that \(p_L^{-1}(U)\) is \(G\)-equivariantly isomorphic to \(U \times \hat{V}\) where \(U = \text{Spec} k[u]_u\).

**Lemma 5.6.** (cf. [9, Lemma 4.2]) The morphism \(p_L : X \to L\) is flat and every closed fiber of \(p_L\) is isomorphic to \(\hat{V}\).

**Proof.** Let \(\bar{p}_L : Y \to L\) be the projection. Then the \(T'\)-action on \(L\) is trivial and \(\bar{p}_L\) is \((G \times T \times T')\)-equivariant. It holds that \(\bar{p}_L = p_L \circ \pi\) where \(\pi : Y \to Y//T' = X\) is the quotient. For a closed point \(c \in L \cong A^1\), \(\bar{p}_L^{-1}(c)\) is \(T'\)-stable and \(p_L^{-1}(c) = \pi(\bar{p}_L^{-1}(c)) = \bar{p}_L^{-1}(c)//T'\) since \(\pi\) is surjective. From the above remark, \(\bar{p}_L^{-1}(c) \cong \hat{V}\) for a closed point \(c \in U\). We show that \(p_L^{-1}(0) \cong V_0 \oplus V_d\). The \(T_i\)-action on \(\bar{p}_L^{-1}(0) = \text{Spec} \hat{B}\) is induced by \(\delta_i\) and its slice \(\hat{w}_i\), where \(\hat{B} = B/(u)\), \(\delta_i\) is the locally nilpotent derivation on \(\hat{B}\) induced by \(\delta_i\), and \(\hat{w}_i\) is the residue class of \(w_i\) in \(\hat{B}\). Since \(\hat{B} = \hat{A}[\hat{w}_1, \ldots, \hat{w}_m]\) where \(\hat{A} = \cap_{i=1}^m \hat{B}^{\delta_i}\), it follows that \(p_L^{-1}(0) = \bar{p}_L^{-1}(0)//T' = \text{Spec} \hat{A}\).

Each derivation \(\delta_i\) on \(\hat{B} \cong k[v_1, \ldots, v_m, x_1, \ldots, x_n]\) is a semi-invariant \(k[v_1, \ldots, v_m]\)-derivation of weight \((-\alpha_i, -d)\) having a slice and satisfies \(\delta_i(x_j) \in k[v_1, \ldots, v_m]\) for \(1 \leq j \leq n\). By Lemma 2.3, it follows that \(\bar{B}^{\delta_i} = k[v_1, \ldots, v_m, \xi_{1,2}, \ldots, \xi_{1,n}]\) where \(\xi_{1,j}\) is of weight \((\beta_j, d)\) and linear in \(x_1, \ldots, x_n\) over \(k[v_1, \ldots, v_m]\). Note that the locally nilpotent derivation \(\delta_2\) on \(\bar{B}^{\delta_1}\) has a slice and \(\delta_2(\xi_{1,j}) \in k[v_1, \ldots, v_m]\) for \(2 \leq j \leq n\). By applying Lemma 2.3 to \(\delta_2\) on \(\bar{B}^{\delta_1}\), we have \(\bar{B}^{\delta_1} \cap \bar{B}^{\delta_2} = k[v_1, \ldots, v_m, \xi_{2,3}, \ldots, \xi_{2,n}]\) where \(\xi_{2,j}\) is of weight \((\beta_j, d)\) and linear in \(x_1, \ldots, x_n\) over \(k[v_1, \ldots, v_m]\). Hence by applying Lemma 2.3 subsequently to \(\delta_i\) on \(\cap_{j=1}^m \bar{B}^{\delta_i}\) for \(2 \leq i \leq m\), we obtain that \(\hat{A}\) is a polynomial ring over \(k[v_1, \ldots, v_m]\) in semi-invariant \(n - m\) variables and \(p_L^{-1}(0) \cong \hat{V}_0 \oplus V_d\). Since every closed fiber \(p_L^{-1}(c)\) is isomorphic to \(A^n\), it follows from [10, 6], [7] that \(p_L\) is flat. \(\square\)

If \(p_L : X \to L\) is an algebraic \(G\)-vector bundle, then \(X\) is isomorphic to a \(G\)-representation. In fact, for a reductive group \(G\), it is well-known that every algebraic \(G\)-vector bundle over a \(G\)-representation \(L\) of dimension one is trivial, i.e., isomorphic to a product bundle \(L \times F\).
for a \(G\)-representation \(F\). Note that \(u\) is a coordinate variable of a polynomial ring \(A\) when \(p_L : X \to L\) is an algebraic \(G\)-vector bundle.

**Proposition 5.7.** Let \(\Delta\) be a sequence having a slice system. Then \(p_L : X \to L\) is an algebraic \((G \times T)\)-vector bundle if and only if the associated \(G\)-embedding \(\varphi_\Delta\) is \(G\)-rectifiable.

**Proof.** Suppose that \(p_L : X \to L\) is an algebraic \((G \times T)\)-vector bundle. Since the fiber \(p_L^{-1}(0)\) is isomorphic to \(V_0 \oplus V'_d\) (cf. the proof of Lemma 5.6), it follows that \(X \cong L \oplus V_0 \oplus V'_d = W\). Hence by Theorem 5.1, \(\varphi_\Delta\) is \(G\)-rectifiable. Conversely, suppose that \(\varphi_\Delta\) is \(G\)-rectifiable. Then by Lemma 4.4, \(A\) is a polynomial ring with semi-invariant coordinates with respect to the \((G \times T)\)-action and \(u\) is a semi-invariant coordinate variable of \(A\). Hence \(p_L : X \to L\) is a trivial \((G \times T)\)-vector bundle.

Finally, we give three examples and discuss problems related to them.

**Example 5.1.** Let \(V = \text{Spec } \mathbb{R}[v]\) and \(\tilde{V} = \text{Spec } \mathbb{R}[x_1, x_2, x_3]\) be \(\mathbb{Z}/2\mathbb{Z}\)-representations with weight 1 and \((1, 0, 1)\), respectively, namely, with the linear \(\mathbb{Z}/2\mathbb{Z}\)-actions

\[
\tau \cdot v = -v, \quad \tau \cdot (x_1, x_2, x_3) = (-x_1, x_2, -x_3)
\]

for a generator \(\tau \in \mathbb{Z}/2\mathbb{Z}\). Let \(\varphi : V \to \tilde{V}\) be an embedding associated with the surjection \(\Phi : \mathbb{R}[x_1, x_2, x_3] \to \mathbb{R}[v]\) defined by \(\Phi(x_1) = v^3, \quad \Phi(x_2) = v^4, \quad \Phi(x_3) = v^5 + v\) (cf. [3]). Then \(\varphi\) is \(\mathbb{Z}/2\mathbb{Z}\)-equivariant. The locally nilpotent derivation \(\delta\) on \(B = \mathbb{R}[u, v, x_1, x_2, x_3]\) associated to \(\varphi\) is

\[
\delta = u^d \partial_u + 3v^2 \partial_{x_1} + 4v^3 \partial_{x_2} + (5v^4 + 1) \partial_{x_3}
\]

which is semi-invariant of weight \((1, -d)\). It is known that \(\varphi\) is rectifiable by [3]. In fact, the \(\mathbb{Z}/2\mathbb{Z}\)-equivariant automorphism \(\Psi\) of \(\mathbb{R}[x_1, x_2, x_3]\) given by

\[
\Psi(x_1) = x_1^3 x_2 + 2x_1^3 x_3 - x_1 x_3^3 \\
\Psi(x_2) = -2 - 5x_2^4 + 6x_1 x_3^5 - 2x_2^2 x_3^6 + (1 + 2x_1 x_3 + 4x_1^2 x_3^2 - 12x_1 x_3^3 + 6x_1^4 x_3^4)(x_2 + 2) + (x_1^4 + 6x_1^5 x_3 - 6x_1^6 x_3^2)(x_2 + 2)^2 + 2x_1^8 (x_2 + 2)^3 \\
\Psi(x_3) = x_1 - (x_1^3 x_2 + 2x_3^3 + x_3 - x_1 x_3^2)^3
\]

rectifies \(\varphi\) into the standard \(\mathbb{Z}/2\mathbb{Z}\)-embedding. Hence the kernel \(A = B^3\) is \((\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^3)\)-equivariantly isomorphic to a polynomial ring \(\mathbb{R}[u, \tilde{v}, y_2, y_3]\) where the weight of \(u, \tilde{v}, y_2\) and \(y_3\) is \((0, -1), (1, 0), (0, d)\) and \((1, d)\), respectively. Therefore \(X = \text{Spec } A\) is a \((\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)\)-representation \(W = \text{Spec } \mathbb{R}[u, \tilde{v}, y_2, y_3]\).
Example 5.2. Let $V^* = \text{Spec } \mathbb{C}[v]$ and $\tilde{V}^* = \text{Spec } \mathbb{C}[x_1, x_2, x_3]$ be the $\mathbb{Z}/4\mathbb{Z}$-representations on which the $\mathbb{Z}/4\mathbb{Z}$-actions are given by

$$\lambda \cdot v = \zeta^v, \quad \lambda \cdot (x_1, x_2, x_3) = (\zeta^3 x_1, x_2, \zeta x_3)$$

where $\lambda$ is a generator of $\mathbb{Z}/4\mathbb{Z}$ and $\zeta$ is the 4-th primitive root of unity. As a representation of the subgroup $\mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}/4\mathbb{Z}$, $V^* = V \otimes_{\mathbb{R}} \mathbb{C} =: V_{\mathbb{C}}$ and $\tilde{V}^* = \tilde{V} \otimes_{\mathbb{R}} \mathbb{C} =: \tilde{V}_{\mathbb{C}}$. The surjection $\Phi : \mathbb{R}[x_1, x_2, x_3] \to \mathbb{R}[v]$ extends to a surjection $\mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[v]$ and the extended surjection defines a $\mathbb{Z}/4\mathbb{Z}$-embedding $\varphi^* : V^* \to \tilde{V}^*$ which restricts to a $\mathbb{Z}/2\mathbb{Z}$-embedding $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to \tilde{V}_{\mathbb{C}}$. The locally nilpotent derivation $\delta_{\mathbb{C}}$ associated to $\varphi^*$ is of the same form as $\delta$ and semi-invariant of weight $(-1, -d)$. Let $A_{\mathbb{C}} = B_{\mathbb{C}}^{\delta_{\mathbb{C}}}$ where $B_{\mathbb{C}} = \mathbb{C}[u, v, x_1, x_2, x_3]$. The automorphism $\Psi$ induces a $\mathbb{Z}/2\mathbb{Z}$-equivariant automorphism $\Psi_{\mathbb{C}}$ of $\mathbb{C}[x_1, x_2, x_3]$, which rectifies $\varphi_{\mathbb{C}}$. Hence $X_{\mathbb{C}} = \text{Spec } A_{\mathbb{C}}$ is isomorphic to the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^*)$-representation $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$. The automorphism $\Psi_{\mathbb{C}}$ is not $\mathbb{Z}/4\mathbb{Z}$-equivariant and we do not know whether the $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{C}^*)$-action on $X \cong A_{\mathbb{A}}^3$ is linearizable or not.

Example 5.3. Let $V$ and $\tilde{V}$ be as in Example 5.1 and let $\varphi : V \to \tilde{V}$ be a $\mathbb{Z}/2\mathbb{Z}$-embedding associated with the surjection $\Phi$ defined by $\Phi(x_1) = v^3 - 3v, \Phi(x_2) = v^4 - 4v^2, \Phi(x_3) = v^5 - 10v$. The locally nilpotent derivation on $B$ associated to $\varphi$ is

$$\delta = u^4 \partial_v + (3v^2 - 3)\partial_{x_1} + (4v^3 - 8v)\partial_{x_2} + (5v^4 - 10)\partial_{x_3}$$

which is semi-invariant of weight $(1, -d)$. It is known that $\varphi$ is non-rectifiable by [11] and [1]. Hence $X$ is not isomorphic to the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-representation $W$. It follows from [1] and [9] that $X$ is not $\mathbb{R}^*$-equivariantly isomorphic to $W$, neither. However, it is not known whether $X$ is isomorphic to $A_{\mathbb{A}}^3$ or not if we forget the action. If $X \cong A_{\mathbb{A}}^3$, then it follows that the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-action on $X \cong A_{\mathbb{A}}^3$ is non-linearizable. By Corollary 5.3, the $(\mathbb{Z}/2\mathbb{Z} \times (\mathbb{R}^*)^2)$-action on $Y \cong A_{\mathbb{R}}^5$ is non-linearizable (cf. [1]). It is unknown neither whether there exists a non-linearizable action of a finite subgroup of $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{R}^*)^2$ on $Y$. The surjection $\Phi$ induces a surjection $\mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[v]$ which defines a $\mathbb{Z}/2\mathbb{Z}$-embedding $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to \tilde{V}_{\mathbb{C}}$. It remains open whether $\varphi_{\mathbb{C}}$ is rectifiable or not.

References


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