EQUIVALENT EMBEDDINGS AND SEMI-INVARIANT LOCALLY NILPOTENT DERIVATIONS

KAYO MASUDA

To Professor R.V. Gurjar on his sixtieth birthday

Abstract. Let G be an abelian reductive algebraic group defined over a field of characteristic 0. Let V and $\tilde{V}$ be G-representation spaces. We assume that $\tilde{V}$ is a direct sum of one-dimensional G-representation spaces. We show that up to the G-equivalence, G-equivariant embeddings $V \to \tilde{V}$ bijectively correspond to the sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems where $m = \dim V$ and δ_i's are semi-invariant, mutually commuting locally nilpotent derivations of some form on a polynomial ring B with a linear $(G \times G_m)$-action. For a sequence Δ as above, the intersection B^{Δ} of the kernels of δ_i for $1 \leq i \leq m$ inherits the $(G \times G_m)$-action on B. We show that B^{Δ} is a polynomial ring with semi-invariant coordinates if and only if the G-equivariant embedding associated to Δ is equivariantly rectifiable. Our results are the equivariant extension of [9].

1. Introduction and results

Let k be a field of characteristic 0, which is the ground field. Let G be an abelian reductive algebraic group. A G-representation space, abbreviated to a G-representation, is an affine space with a linear G-action as a G-variety. Let V and $\tilde{V}$ be G-representations of dimension m and n, respectively. We assume that $\tilde{V}$ is a direct sum of one-dimensional G-representations. Hence the coordinate ring of $\tilde{V}$ is a polynomial ring with semi-invariant coordinates. Let $\varphi : V \hookrightarrow \tilde{V}$ be a G-equivariant embedding of G-varieties, which we abbreviate to a G-embedding. We may assume that φ maps the origin of V to the origin of $\tilde{V}$. Since φ induces a G-equivariant injective homomorphism of the tangent space at the origin of V into the tangent space at the origin of

2000 *Mathematics Subject Classification.* Primary: 14R10; Secondary: 14R20, 14R25.

Key words and phrases. Locally nilpotent derivation, equivariant embedding, Linearization Problem.

Supported by Grant-in-Aid for Scientific Research (C) 18540045, JSPS.

$\tilde{V}$, it follows that $\tilde{V} = V \oplus V'$ for some G-representation V'. Hence V is a direct sum of one-dimensional G-representations as well. For two G-embeddings φ and φ' of V into $\tilde{V}$, φ' is called G-equivalent to φ if there is a G-equivariant automorphism γ of $\tilde{V}$ such that $\varphi' = \gamma \circ \varphi$. A G-embedding $\varphi : V \to \tilde{V}$ is called *G-rectifiable* if there exists a system of semi-invariant coordinate functions $f_1, \ldots, f_n$ on $\tilde{V}$ such that the image $\varphi(V)$ is defined by the G-stable ideal $(f_{m+1}, \ldots, f_n)$. If $\varphi : V \to \tilde{V}$ is G-equivalent to the standard G-embedding $V \hookrightarrow V \oplus V' = \tilde{V}$, then φ is G-rectifiable. Forgetting the G-action, a G-rectifiable embedding $V \to \tilde{V}$ is a rectifiable embedding $\mathbb{A}^m \to \mathbb{A}^n$. It is known by van den Essen and van Rossum [5] that there is a sequence of locally nilpotent derivations associated to a given embedding $\mathbb{A}^m \to \mathbb{A}^n$. In fact, the sequences of locally nilpotent derivations of some form bijectively correspond to embeddings up to the equivalence ([9]). In this article, we show that the same holds true equivariantly.

We fix the notation and state our results. Let Ω be the set of characters of G. Let $V = \operatorname{Spec} k[v_1, \ldots, v_m]$ and let v_i $(1 \leq i \leq m)$ be a semi-invariant of weight $\alpha_i \in \Omega$, i.e., $g \cdot v_i = \alpha_i(g) v_i$ for all $g \in G$. Let $\beta_1, \ldots, \beta_n \in \Omega$ be the weights of a system of semi-invariant coordinate functions of $\tilde{V}$. Let B be a polynomial ring $k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]$ with a linear G-action such that the weight of u, v_i $(1 \leq i \leq m)$, x_j $(1 \leq j \leq n)$ is 0, α_i, β_j, respectively. As a G-variety, $Y = \operatorname{Spec} B$ is isomorphic to $L \oplus V \oplus \tilde{V}$ where L is the trivial G-representation of one dimension. We give B a $\mathbb{Z}$-grading by

$$\deg u = -1, \quad \deg v_i = 0, \quad \deg x_j = d$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$ where d is a fixed positive integer. There is an algebraic action of an algebraic torus G_m on Y corresponding to this grading, which we call the T-action where $T = G_m$. The T-action on Y commutes with the G-action. Note that the algebraic quotient $Y//T = \operatorname{Spec} B^T$ is G-equivariantly isomorphic to $V \oplus \tilde{V}$ where B^T is the T-invariants of B. It follows that $B = \oplus_{(\chi,i) \in \Omega \oplus \mathbb{Z}} B_{\chi,i}$ where

$$B_{\chi,i} = \{b \in B \mid (g,t) \cdot b = t^i \chi(g) b \quad \text{for all } (g,t) \in G \times T\}.$$

A derivation δ on $B = \oplus_{(\chi,i) \in \Omega \oplus \mathbb{Z}} B_{\chi,i}$ is called semi-invariant of weight (ω, ℓ) if $\delta(B_{\chi,i}) \subset B_{\chi+\omega, i+\ell}$ for every (χ, i). We consider a sequence $\Delta = (\delta_1, \ldots, \delta_m)$ of semi-invariant, locally nilpotent derivations δ_i's on B satisfying $\delta_i \delta_l = \delta_l \delta_i$ for any i and l. For such two sequences $\Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)})$ and $\Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)})$, we say that Δ_1 and Δ_2 are G-equivalent if there is a $(G \times T)$-equivariant $k[v_1, \ldots, v_m]$-automorphism ψ of B which satisfies $\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)}$ for every i. We say that the sequence $\Delta = (\delta_1, \ldots, \delta_m)$ of mutually commuting locally nilpotent derivations on B has a slice system $(s_1, \cdots, s_m)$ if each δ_i has a slice $s_i \in B$, i.e., $\delta_i(s_i) = 1$, such that $\delta_i(s_l) = 0$ for $i \neq l$. When the G-action is trivial, it is shown in [9] that there is a bijective correspondence between the equivalence classes of embeddings $\mathbb{A}^m \to \mathbb{A}^n$ and the equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems where δ_i's are mutually commuting locally nilpotent derivations on B and semi-invariant of weight $-d$ of a form

$$\delta_i = u^d \partial_{v_i} + f_{i1}\partial_{x_1} + \cdots + f_{in}\partial_{x_n} \qquad (*)$$

for $f_{ij} \in B^T$. For a sequence $\Delta = (\delta_1, \ldots, \delta_m)$ as above, let $B^\Delta = \cap_{i=1}^m B^{\delta_i}$ where B^{δ_i} denotes the kernel of δ_i. Then B^Δ inherits the T-action on B. It is known as well that B^Δ is a polynomial ring with semi-invariant coordinates with respect to the T-action if and only if the embedding $\varphi_\Delta : \mathbb{A}^m \to \mathbb{A}^n$ associated to Δ is rectifiable ([9]). Throughout this article, a sequence $\Delta = (\delta_1, \ldots, \delta_m)$ implies a sequence of mutually commuting locally nilpotent derivations δ_i's on B such that each δ_i is of the form $(*)$ and semi-invariant of weight $(-\alpha_i, -d)$ for $1 \leq i \leq m$ unless otherwise stated. The next result follows from Proposition 4.3 and Lemma 4.6, which appear below.

Theorem 1.1. *There is a bijective correspondence between the G-equivalence classes of G-embeddings $V \to \tilde{V}$ and the G-equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ having slice systems.*

The kernel B^Δ of a sequence Δ inherits the $(G \times T)$-action on B. We show the following (Theorem 5.1).

Theorem 1.2. *Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence having a slice system. Then B^Δ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action if and only if the G-embedding $\varphi_\Delta : V \to \tilde{V}$ associated to Δ is G-rectifiable.*

Suppose that a G-embedding $\varphi : V \to \tilde{V}$ is rectifiable, but not G-rectifiable. Then $X = \text{Spec } B^\Delta$ is isomorphic to a T-representation, hence the affine space of dimension $n+1$, but not isomorphic to a $(G \times T)$-representation. Namely, the $(G \times T)$-action on $X \cong \mathbb{A}^{n+1}$ is non-linearizable.

We observe the G-rectifiability of G-embeddings in section 3. In section 5, we observe an example of non-rectifiable $\mathbb{Z}/2\mathbb{Z}$-embedding and discuss problems related to it. In general, it is not easy to determine whether the kernel B^Δ of a sequence Δ having a slice system is a polynomial ring or not. However, there are some cases that B^Δ is a polynomial ring with semi-invariant coordinates, i.e., $X = \text{Spec } B^\Delta$

is isomorphic to a representation of G or $G \times T$. We give a couple of examples. Suppose that the ground field k is algebraically closed. The G-action on an affine variety Y is called *fix-pointed* if every closed orbit of Y is a fixed point. Suppose that the G-action on Spec B is fix-pointed and $V^G = \{0\}$. Then B^Δ is a polynomial ring with semi-invariant coordinates with respect to the G-action (Proposition 5.4). Consider, next, the case where G is a complex algebraic torus and V and $\tilde{V}$ are complex G-representations. Suppose that $\dim \tilde{V}/\!/G \leq 1$. Then it follows from the results of Bass and Haboush [2] and Kraft and Schwarz [8] that B^Δ is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action (Theorem 5.5). Hence by Theorems 1.1 and 1.2, it follows that every G-embedding $V \to \tilde{V}$ is G-rectifiable in this case.

Our results hold for a reductive group G not necessarily abelian under the assumption that $\tilde{V}$ is a direct sum of one-dimensional G-representations unless otherwise G is specified.

Acknowledgements The author is grateful to the referee for making suggestions for improving the presentation.

2. Preliminaries

We collect some results on the kernel of a sequence of semi-invariant, mutually commuting locally nilpotent derivations on an affine domain with an action of an abelian reductive group.

Let G be an abelian reductive algebraic group and let B be an affine domain with an algebraic G-action. The subalgebra B^G of G-invariants is finitely generated over k. Let Ω be the set of characters of G. We assume that B has a decomposition $B = \oplus_{\chi \in \Omega} B_\chi$ where

$$B_\chi = \{b \in B \mid g \cdot b = \chi(g) b \quad \text{for all } g \in G\}.$$

An element of B_χ is called semi-invariant of weight χ.

Suppose that δ is a locally nilpotent derivation on B which is semi-invariant of weight $-\omega$ and that δ has a slice $s \in B$. Then we may assume that s is semi-invariant of weight ω.

Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence of mutually commuting locally nilpotent derivations on $B = \oplus_{\chi \in \Omega} B_\chi$. Suppose that δ_i is semi-invariant of weight $-\omega_i$ for $1 \leq i \leq m$ and that Δ has a semi-invariant slice system $(s_1, \ldots, s_m)$ such that s_i is of weight ω_i.

Lemma 2.1. (cf. [9, Lemma 2.1]) *Under the notation and assumption above, we have the following.*

(1) *The slice* s_i *is transcendental over* B^{δ_i} *and* $B = B^{\delta_i}[s_i]$. *Hence*

$$B = A[s_1, \ldots, s_m] \qquad \textit{for} \quad A = \bigcap_{i=1}^{m} B^{\delta_i}.$$

(2) *Let* $\pi_{s_i} : B \to B$ *be the Dixmier map induced by* δ_i *and* s_i, *which is the algebra homomorphism defined by*

$$\pi_{s_i}(b) = \sum_{j \geq 0} \frac{(-1)^j}{j!} \delta_i^j(b) s_i^j \qquad \textit{for} \quad b \in B.$$

Then

(i) $\pi_{s_i}(B_\chi) \subset B_\chi$ *for every* $\chi \in \Omega$.

(ii) $B^{\delta_i} = \pi_{s_i}(B)$.

(iii) *The kernel of* π_{s_i} *is the* G*-stable ideal* $(s_i) \subset B$.

Hence $A = \pi_s(B)$ *where* $\pi_s = \pi_{s_m} \circ \cdots \circ \pi_{s_1}$ *and* A *inherits the* G*-action.*

The sequence $\Delta = (\delta_1, \cdots, \delta_m)$ on B having a semi-invariant slice system induces a $\mathbb{Z}^m$-grading on B, hence an algebraic action of an m-dimensional torus $(G_m)^m$ on $Y := \operatorname{Spec} B$. In fact, for $1 \leq i \leq m$, δ_i and its slice s_i induce a $\mathbb{Z}$-grading on B such that $\deg s_i = 1$ and $\deg a = 0$ for $a \in B^{\delta_i} - \{0\}$. There is an algebraic G_m-action on Y corresponding to the $\mathbb{Z}$-grading on B induced by δ_i and s_i. We call the action the T_i-action where $T_i = G_m$. The T_i-action on Y is given by the k-algebra homomorphism $\rho_t : B \to B[t, t^{-1}]$ defined by (cf. Freudenburg [6, 10.2])

$$\rho_t(b) = \sum_{j \geq 0} \frac{(t-1)^j}{j!} \delta_i^j(b) s_i^j \qquad \text{for} \quad b \in B.$$

The T_i-actions commute with each other and the m-dimensional torus $T' = T_1 \times \cdots \times T_m$ acts on Y. The subalgebra B^{T_i} is equal to $B^{\delta_i} \cong B/(s_i)$ and $B^{T'} = A$. Note that the G-action commutes with every T_i-action and $G \times T'$ acts on Y.

For the remainder of this section, we suppose that k is algebraically closed. Let $p : Y \to Y/\!/G := \operatorname{Spec} B^G$ be the algebraic quotient, which is defined by the inclusion $B^G \hookrightarrow B$. Then p induces an embedding of the fixed point locus Y^G into $Y/\!/G$. The G-action on Y is called *fix-pointed* if $p|_{Y^G}$ is an isomorphism, i.e., every closed orbit of Y is a fixed point. It is known by Bass and Haboush [2] that $Y \cong Y^G \times V$ for a G-representation V if Y is smooth, the G-action on Y is fix-pointed, and every vector bundle over Y is trivial. Hence if $Y^G \cong Y/\!/G$ is the affine space, then Y is isomorphic to the affine space and the G-action on Y is linearizable.

Lemma 2.2. *Under the notation and assumption in Lemma 2.1, suppose that $Y = \mathrm{Spec}\, B$ is smooth and the G-action on Y is fix-pointed. Suppose, further, that every vector bundle over Y is trivial. Then A is a polynomial ring over A^G with semi-invariant coordinates. Hence if A^G is a polynomial ring, then A is a polynomial ring over k with semi-invariant coordinates, i.e., $X = \mathrm{Spec}\, A$ is isomorphic to a G-representation.*

Proof. By Lemma 2.1, it follows that $Y \cong X \times V$ for a G-representation V. Hence X is a smooth G-subvariety of Y. Note that the G-action on X is fix-pointed as well. Further, every vector bundle over X is trivial since every vector bundle over Y is trivial. Hence it follows from [2] that $X \cong X//G \times V'$ for a G-representation V'. □

If the G-action on Y is fix-pointed, then the G-action on $X_i = \mathrm{Spec} A_i$ is fix-pointed as well where $A_i = B^{\delta_i}$.

If $G = G_m$, then B is $\mathbb{Z}$-graded and $B = \oplus_{i \in \mathbb{Z}} B_i$. Suppose that B is positively graded, i.e., $B_i = 0$ for every $i < 0$. Then the G-action on Y is fix-pointed. Hence A is a polynomial ring with homogeneous coordinates if A^G is a polynomial ring.

Let B be an affine domain with a G-action and let $R = B[x_1, \ldots, x_n]$ be a polynomial ring over B with a G-action where an inderminant x_j is semi-invariant of weight β_j. We assume that the G-action on R restricts to the action on B. Let δ be a semi-invariant locally nilpotent B-derivation on R of weight $-\beta_1$ and let $f_j = \delta(x_j) \in B$ for $1 \leq j \leq n$. Then each f_j is semi-invariant of weight $\beta_j - \beta_1$. Suppose that δ has a slice. Then the sequence $(f_1, \ldots, f_n)$ is a unimodular row. Hence the sequence $(f_1, \ldots, f_n)$ is considerd as a surjective G-vector bundle map Ψ from the trivial G-bundle $Y \times (L \oplus F)$ over $Y = \mathrm{Spec} B$ onto the trivial G-bundle $Y \times L$ where L is the trivial G-representation of dimension one and F is a G-representation of weight $\beta_2 - \beta_1, \ldots, \beta_n - \beta_1$. In fact, Ψ is given by

$$\Psi(y, c_1, \ldots, c_n) = (y, f_1(y)c_1 + \cdots + f_n(y)c_n)$$

for $y \in Y$ and $(c_1, \ldots, c_n) \in L \oplus F$ (The G-action on B is given by $(g \cdot f)(y) = f(g^{-1}y)$ for $g \in G$, $f \in B$, and $y \in Y$). Suppose, further, that any G-vector bundle over Y is trivial. Then the G-vector bundle obtained by the kernel of Ψ can be trivialized. Hence there exists a G-vector bundle automorphism of $Y \times (L \oplus F)$, which is represented by an invertible matrix C of entries in semi-invariants of B such that $(f_1, \ldots, f_n) = (1, 0, \ldots, 0)C$.

Lemma 2.3. (cf. [9, Lemma 2.2]) *Let δ be a semi-invariant locally nilpotent B-derivation on $R = B[x_1, \ldots, x_n]$ of weight $-\beta_1$ having a* *slice and let* $\delta(x_j) \in B$ *for every* j. *Suppose that any* G*-vector bundle over* Y *is trivial. Then* R^δ *is a polynomial ring over* B. *Furthermore, there exists a system of coordinates* $\xi_2, \dots, \xi_n$ *of* R^δ *over* B *such that each* ξ_i *is semi-invariant of weight* β_i *and linear in* $x_1, \dots, x_n$ *over* B.

Proof. Let $f_j = \delta(x_j)$ for $1 \leq j \leq n$. Then there is an invertible matrix C of entries in semi-invariants of B such that $(f_1, \dots, f_n) = (1, 0, \dots, 0)C$. Let $D = (d_{ij})_{1\leq i,j\leq n}$ be the inverse of C. Then $d_{ij} \in B$ is semi-invariant of weight $\beta_j - \beta_i$ since C represents a G-vector bundle automorphism of $Y \times (L \oplus F)$. Let

$$\begin{aligned} s &= d_{11}x_1 + d_{21}x_2 + \cdots + d_{n1}x_n \\ \xi_i &= d_{1i}x_1 + d_{2i}x_2 + \cdots + d_{ni}x_n \qquad \text{for} \quad 2 \leq i \leq n. \end{aligned}$$

Then $R = B[s, \xi_2, \dots, \xi_n]$ and s is of weight β_1 and ξ_i of weight β_i. Since $(f_1, \dots, f_n)D = (1, 0, \dots, 0)$, it follows that s is a slice of δ and $\delta(\xi_i) = 0$ for $2 \leq i \leq n$. Hence $R^\delta = B[\xi_2, \dots, \xi_n]$ and the assertion follows. □

The following is easily verified.

Lemma 2.4. (cf. [9, Lemma 2.3]) *Let* δ *be a semi-invariant locally nilpotent* B*-derivation on* $R = B[x_1, \dots, x_n]$. *Suppose that* $\delta(x_1)$ *belongs to the group* B^* *of invertible elements of* B. *Then* $R^\delta = B[x_2, \dots, x_n]$. *Hence* R^δ *is a polynomial ring over* B *with semi-invariant coordinates.*

3. Equivariant embeddings

In this section, we observe G-embeddings and associate a sequence to a given G-embedding.

Let V and $\tilde{V}$ be G-representations and let $\varphi : V \to \tilde{V}$ be a G-embedding. Let $S = k[v_1, \dots, v_m]$ and $R = k[X_1, \dots, X_n]$ be the coordinate rings of V and $\tilde{V}$ respectively, where v_i ($1 \leq i \leq m$) is semi-invariant of weight α_i and X_j ($1 \leq j \leq n$) of weight β_j. As observed in section 1, it follows that $\tilde{V} = V \oplus V'$ for a G-representation V'. Hence we may and assume $\beta_i = \alpha_i$ for $1 \leq i \leq m$.

Proposition 3.1. *Suppose that there exists a subgroup* G' *of* G *such that* $\tilde{V}^{G'} = V$. *Then* $\varphi : V \to \tilde{V}$ *is* G*-rectifiable.*

Proof. Note that $V^{G'} = V$ since $V = \tilde{V}^{G'}$ is G'-invariant. Hence the G-embedding φ induces an embedding $V^{G'} \to \tilde{V}^{G'}$, which is an isomorphism. Thus φ is G-rectifiable. □

Let $\Phi : R \to S$ be the G-equivariant surjection associated with the G-embedding $\varphi : V \to \tilde{V}$. Since Φ is G-equivariant, Φ induces a surjection $\Phi_\chi : R_\chi \to S_\chi$ for every character χ of G.

Proposition 3.2. (1) *Suppose that for $1 \leq i \leq m$, α_i is non-trivial and the R^G-module R_{α_i} is generated by one element. Then φ is G-rectifiable.*

(2) *Suppose that $S_{\beta_j} = 0$ for $m+1 \leq j \leq n$. Then φ is G-rectifiable.*

Proof. (1) It suffices to show that $\Phi(X_i) = c_i v_i$ for $1 \leq i \leq m$ where $c_i \in k^*$. In fact, if $\Phi(X_i) = c_i v_i$ with $c_i \in k^*$ for $1 \leq i \leq m$, then we have a system of semi-invariant coordinates $X_1, \ldots, X_m, X'_{m+1}, \ldots, X'_n$ of R satisfying $\Phi(X'_{m+j}) = 0$ for $1 \leq j \leq n-m$ where $X'_{m+j} = X_{m+j} - f_{m+j}(c_1^{-1}X_1, \cdots, c_m^{-1}X_m)$ for $f_{m+j} = \Phi(X_{m+j})$. It follows from the assumption that for $1 \leq i \leq m$, R_{α_i} is generated by X_i over R^G since X_i is irreducible. Since $\Phi_{\alpha_i} : R_{\alpha_i} \to S_{\alpha_i}$ is surjective, there exists $f_i \in R^G$ such that $\Phi_{\alpha_i}(f_i X_i) = \Phi(f_i)\Phi(X_i) = v_i$. It follows that $\Phi(f_i) \in S^* = k^*$ since v_i is irreducible and semi-invariant of non-trivial weight. Hence $\Phi(X_i) = c_i v_i$ for $c_i \in k^*$.

(2) Since $\Phi_{\beta_j} : R_{\beta_j} \to S_{\beta_j}$ is trivial, $\Phi(X_j) = 0$ for $m+1 \leq j \leq n$. Hence the assertion follows. □

We show that every G-embedding $V \to \tilde{V}$ is G-rectifiable when composed by a suitable G-embedding $\tilde{V} \to V \oplus \tilde{V}$.

Proposition 3.3. *There exists a G-embedding $\psi : \tilde{V} \to V \oplus \tilde{V}$ such that $\tilde{\varphi} = \psi \circ \varphi$ is G-rectifiable.*

Proof. Since $\Phi : R \to S$ is a G-equivariant surjection, there exists an $H_i \in R$ of weight α_i such that $\Phi(H_i) = v_i$ for $1 \leq i \leq m$. Let I be the kernel of Φ. Then I is G-stable and $R = k[H_1, \ldots, H_m] \oplus I$ as a G-module. Hence each X_j is written as $X_j = \xi_j + \eta_j$ where $\xi_j \in k[H_1, \ldots, H_m]$ and $\eta_j \in I$ are of weight β_j. Thus the semi-invariant $m+n$ elements $H_1, \ldots, H_{m+n}$ generate R over k, where $H_{m+j} := \eta_j$ for $1 \leq j \leq n$. Note that $I = (H_{m+1}, \ldots, H_{m+n})$. Let $\tilde{R} = k[\tilde{X}_1, \ldots, \tilde{X}_{m+n}]$ be a polynomial ring with a linear G-action such that the weight of $\tilde{X}_j$ is α_j for $1 \leq j \leq m$ and β_{j-m} for $m+1 \leq j \leq m+n$. Then Spec $\tilde{R}$ is isomorphic to the G-representation $V \oplus \tilde{V}$. Let $\Psi : \tilde{R} \to R$ be the G-equivariant surjection defined by $\Psi(\tilde{X}_j) = H_j$ for $1 \leq j \leq m+n$. Then Ψ defines a G-embedding $\psi : \tilde{V} \to V \oplus \tilde{V}$ and the kernel of the G-equivariant surjection $\tilde{\Phi} := \Phi \circ \Psi$ is the G-stable ideal generated by $\tilde{X}_{m+1}, \ldots, \tilde{X}_{m+n}$. Hence the G-embedding $\tilde{\varphi} = \psi \circ \varphi : V \to V \oplus \tilde{V}$ is G-rectifiable and the assertion follows. □

We shall associate a sequence Δ_φ to the G-embedding $\varphi : V \to \tilde{V}$. Let B be a polynomial ring $k[u, v_1, \dots, v_m, x_1, \dots, x_n]$ with a linear $(G \times T)$-action where $T = G_m$ as stated in section 1. The weight of u, v_i $(1 \le i \le m)$, x_j $(1 \le j \le n)$ is $(0, -1)$, $(\alpha_i, 0)$, (β_j, d), respectively, where d is a positive integer. Let f_j be the image of X_j by $\Phi : R \to S$ for $1 \le j \le n$. We define a derivation δ_i on B by

$$\delta_i = u^d \partial_{v_i} + (\partial_{v_i} f_1)\partial_{x_1} + \cdots + (\partial_{v_i} f_n)\partial_{x_n}$$

for $1 \le i \le m$. Then δ_i is a semi-invariant locally nilpotent derivation of weight $(-\alpha_i, -d)$ and $\delta_i \delta_l = \delta_l \delta_i$ for any i and l. Let $\Delta_\varphi = (\delta_1, \dots, \delta_m)$.

Proposition 3.4. ([9, Proposition 3.2]) *The sequence $\Delta_\varphi = (\delta_1, \dots, \delta_m)$ has a semi-invariant slice system $(s_1, \dots, s_m)$.*

Proof. We show that each δ_i has a slice s_i of weight (α_i, d) such that $\delta_l(s_i) = 0$ for $i \ne l$. Since Φ is a surjection, there exists a polynomial $F_i \in k[X_1, \dots, X_n]$ such that

$$F_i(f_1, \dots, f_n) = v_i$$

for $1 \le i \le m$. Let $s_i \in B$ be an element satisfying

$$F_i(f_1 - u^d x_1, \dots, f_n - u^d x_n) = v_i - u^d s_i.$$

Since $\delta_l(f_j - u^d x_j) = 0$ for $1 \le l \le m$ and $1 \le j \le n$, it follows that $\delta_l(v_i - u^d s_i) = 0$, i.e., $\delta_i(s_i) = 1$ and $\delta_l(s_i) = 0$ for $i \ne l$. We can take s_i to be semi-invariant of weight (α_i, d), and the assertion follows. $\square$

We call Δ_φ the sequence associated to a G-embedding φ.

4. Sequences and G-embeddings

In this section, we show that there is a bijective correspondence between the G-equivalence classes of G-embeddings and the G-equivalence classes of sequences having slice systems.

Let

$$B = k[u, v_1, \dots, v_m, x_1, \dots, x_n]$$

where the weight of u, v_i, x_j is $(0, -1)$, $(\alpha_i, 0)$, (β_j, d), respectively, as in the previous section. We assume that $\beta_i = \alpha_i$ for $1 \le i \le m$. Note that

$$B^T = k[v_1, \dots, v_m, u^d x_1, \dots, u^d x_n].$$

Let $Y = \operatorname{Spec} B$. As a $(G \times T)$-representation,

$$Y = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d}.$$

Here $L = \operatorname{Spec} k[u]$, $V_0 = \operatorname{Spec} S$, $V_{-d} = \operatorname{Spec} k[x_1, \dots, x_m]$, and $V'_{-d} = \operatorname{Spec} k[x_{m+1}, \dots, x_n]$ where $S = k[v_1, \dots, v_m]$. As a G-representation,

$V_0 = V = V_{-d}$. We shall observe the kernel of a sequence. Let $\Delta = (\delta_1, \dots, \delta_m)$ be a sequence, i.e., a sequence of mutually commuting locally nilpotent derivations on B of a form

$$\delta_i = u^d \partial_{v_i} + f_{i1}\partial_{x_1} + \cdots + f_{in}\partial_{x_n} \tag{1}$$

where $f_{ij} \in B$ is semi-invariant of weight $(\beta_j - \alpha_i, 0)$ for $1 \le j \le n$. Each δ_i is semi-invariant of weight $(-\alpha_i, -d)$. Let

$$A = \bigcap_{i=1}^{m} B^{\delta_i}.$$

The sequence $\Delta = (\delta_1, \dots, \delta_m)$ uniquely extends to a sequence of mutually commuting semi-invariant locally nilpotent derivations on the localization B_u at u. Note that the $(G \times T)$-action on B extends to the action on B_u. The kernel $(B_u)^{\delta_i}$ is the localization of B^{δ_i} at u, and $\cap_{i=1}^m (B_u)^{\delta_i} = A_u$. The sequence $\Delta = (\delta_1, \dots, \delta_m)$ of the derivations on B_u has a semi-invariant slice system $(v_1/u^d, \dots, v_m/u^d)$. Hence it follows from Lemma 2.1 that $B_u = A_u[v_1, \dots, v_m]$ and

$$A_u = k[u, u^{-1}, \phi(x_1), \dots, \phi(x_n)]$$

where $\phi = \phi_m \circ \cdots \circ \phi_1$ and ϕ_i is the Dixmier map defined by

$$\phi_i(b) = b + \sum_{l \ge 1} \frac{(-1)^l}{l!} \delta_i^l(b) \left(\frac{v_i}{u^d}\right)^l \qquad \text{for} \quad b \in B_u.$$

Note that each ϕ_i preserves the weight since δ_i is semi-invariant of weight $(-\alpha_i, -d)$ and v_i/u^d is of weight (α_i, d). The following is easily proved (cf. [9]).

Lemma 4.1. (1) *For $1 \le j \le n$, $u^d\phi(x_j)$ is uniquely written as*

$$u^d\phi(x_j) = u^d\theta_j + h_j$$

where $\theta_j \in B$ is an element of weight (β_j, d) satisfying $u^d\theta_j \in B^T \cap u^d B$ and $h_j \in S$ is of weight $(\beta_j, 0)$.

(2)

$$\begin{aligned} A^T &= k[u^d\phi(x_1), \dots, u^d\phi(x_n)] \\ &= k[u^d\theta_1 + h_1, \dots, u^d\theta_n + h_n]. \end{aligned}$$

Hence A^T is a polynomial ring in semi-invariant n variables.

Note that Spec A^T is G-equivariantly isomorphic to $\tilde{V} = \operatorname{Spec} R$.

Let $\bar{B} = B/(u) \cong k[v_1, \dots, v_m, x_1, \dots, x_n]$. Then $\bar{B}$ inherits the $(G \times T)$-action and the surjection $q : B \to \bar{B}$ preserves the weight. The derivation δ_i induces a semi-invariant locally nilpotent derivation

$\bar{\delta}_i$ on $\bar{B}$. Let $\bar{A} = \cap_{i=1}^m \bar{B}^{\bar{\delta}_i}$. Since $q \circ \delta_i = \bar{\delta}_i \circ q$ for every i, q induces an algebra homomorphism $q_0 : A \to \bar{A}$ which preserves the weight.

Lemma 4.2. ([9, Lemma 3.1]) *The following are equivalent.*

(1) *The sequence Δ has a slice system.*
(2) *The sequence Δ has a semi-invariant slice system $(s_1, \dots, s_m)$ such that s_i is of weight (α_i, d).*
(3) *The homomorphism $q_0 : A \to \bar{A}$ is surjective.*

In the following, a semi-invariant slice system $(s_1, \dots, s_m)$ of Δ implies that $(s_1, \dots, s_m)$ is a slice system of Δ such that s_i is semi-invariant of weight (α_i, d) for $1 \le i \le m$. Suppose that $\Delta = (\delta_1, \dots, \delta_m)$ has a semi-invariant slice system $(s_1, \dots, s_m)$. Then $v_i - u^d s_i \in A^T$ for every i. By Lemma 4.1 (2), there exists a polynomial $F_i \in k[X_1, \dots, X_n]$ for $1 \le i \le m$ such that

$$v_i - u^d s_i = F_i(u^d\theta_1 + h_1, \dots, u^d\theta_n + h_n).$$

Substituting $u = 0$ to the above equation, we obtain

$$v_i = F_i(h_1, \dots, h_n). \tag{2}$$

We consider $R = k[X_1, \dots, X_n]$ as a polynomial ring with a linear $(G \times T)$-action such that X_j is of weight $(\beta_j, 0)$ for $1 \le j \le n$. As a G-representation, Spec $R = \tilde{V}$. We define a G-equivariant algebra homomorphism $\Phi_\Delta : R \to S$ by

$$\Phi_\Delta(X_j) = h_j \qquad \text{for} \quad 1 \le j \le n.$$

Since Φ_Δ is a surjection by (2), it defines a G-embedding $\varphi_\Delta : V \hookrightarrow \tilde{V}$. In particular, we have $n \ge m$. We call φ_Δ (resp. Φ_Δ) the G-embedding (resp. G-equivariant surjection) associated to Δ.

Proposition 4.3. *Let $\varphi : V \to \tilde{V}$ be a G-embedding and let $\Delta_\varphi = (\delta_1, \dots, \delta_m)$ be the sequence associated to φ. Then the G-embedding φ_{Δ_φ} associated to Δ_φ is G-equivalent to φ.*

Proof. Let $\Phi : R \to S$ be the G-equivariant surjection associated with φ. Recall that δ_i of Δ_φ is defined by

$$\delta_i = u^d \partial_{v_i} + (\partial_{v_i} f_1)\partial_{x_1} + \cdots + (\partial_{v_i} f_n)\partial_{x_n}$$

where $f_j = \Phi(X_j)$ for $1 \le j \le n$. The associated surjection $\Phi_{\Delta_\varphi} : R \to S$ is defined by $\Phi_{\Delta_\varphi}(X_j) = h_j$ where $h_j \in S$ is a polynomial satisfying $u^d \phi(x_j) = u^d\theta_i + h_j$ for $u^d\theta_i \in B^T$. It is shown in [9, Proposition 3.2] that $h_j = -f_j + c_j$ for some $c_j \in k$. Since both of the weight of h_j and of f_j are equal to $(\beta_j, 0)$, it follows that $c_j = 0$ if β_j is non-trivial.

Hence by a G-equivariant affine automorphism γ of R, it follows that $\Phi_{\Delta_\varphi} = \Phi \circ \gamma$, i.e., φ_{Δ_φ} is G-equivalent to φ. □

Suppose that $\Delta = (\delta_1, \dots, \delta_m)$ has a slice system. Then the associated G-embedding φ_Δ, equivalently, the G-equivariant surjection $\Phi_\Delta : R \to S$ can be defined. Since Φ_Δ is a surjection, there exists an $H_i \in R$ of weight α_i such that $\Phi_\Delta(H_i) = v_i$ for $1 \le i \le m$. Let I be the kernel of Φ_Δ. Then I is G-stable and $R = k[H_1, \dots, H_m] \oplus I$ as a G-module. There is a set of semi-invariant generators $H_{m+1}, \dots, H_{m+n}$ of I such that $H_1, \dots, H_{m+n}$ generate R over k (cf. the proof of Proposition 3.3). We can choose appropriate generators among $H_{m+1}, \dots, H_{m+n}$ and change their subscripts if necessary so that $I = (H_{m+1}, \dots, H_r)$ and $H_1, \dots, H_m, H_{m+1}, \dots, H_r$ generate R. Note that $r \ge n$. The weight of H_i is γ_i where $\gamma_i = \alpha_i$ if $1 \le i \le m$ and $\gamma_i = \beta_j$ for some j if $m + 1 \le i \le r$. We have

$$\Phi_\Delta(H_i) = H_i(h_1, \cdots, h_n) = \begin{cases} v_i & \text{for} \quad 1 \le i \le m \\ 0 & \text{for} \quad m+1 \le i \le r. \end{cases}$$

Note that if φ_Δ is G-rectifiable, then we can take $r = n$ and $H_1, \dots, H_n$ is a system of semi-invariant coordinates of R.

Let

$$\sigma : R \to A^T = k[u^d\theta_1 + h_1, \dots, u^d\theta_n + h_n]$$

be the G-equivariant isomorphism defined by $X_i \mapsto u^d\theta_i + h_i$. We define w_i $(1 \le i \le m)$ and y_j $(m + 1 \le j \le r)$ by

$$\begin{aligned} \sigma(H_i) &= H_i(u^d\theta_1 + h_1, \dots, u^d\theta_n + h_n) \\ &= \begin{cases} v_i - u^d w_i & \text{for} \quad 1 \le i \le m \\ u^d y_i & \text{for} \quad m+1 \le i \le r. \end{cases} \end{aligned}$$

Since $\sigma(H_i) \in A^T$, it follows that for every l

$$\delta_l(v_i - u^d w_i) = \delta_l(u^d y_j) = 0.$$

Hence $(w_1, \dots, w_m)$ is a semi-invariant slice system of Δ, and y_j $(m + 1 \le j \le r)$ is an element of A of weight (γ_j, d). Since $H_1, \dots, H_r$ generate R, it follows from $A^T = \sigma(R)$ that

$$A^T = k[\tilde{v}_1, \dots, \tilde{v}_m, u^d y_{m+1}, \dots, u^d y_r]$$

where $\tilde{v}_i := v_i - u^d w_i$ for $1 \le i \le m$.

Lemma 4.4. (cf. [9, Lemmas 3.3 and 3.5]) *Suppose that* $\Delta = (\delta_1, \dots, \delta_m)$ *has a slice system.*

(1)
$$A = k[u, \tilde{v}_1, \dots, \tilde{v}_m, y_{m+1}, \dots, y_r].$$

As a consequence, if the associated G-embedding φ_Δ is G-rectifiable, then A is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action.

(2)
$$A = k[u, u^d\theta_1 + h_1, \cdots, u^d\theta_n + h_n, u^{-d}I(u^d\theta + h)]$$
where $I(u^d\theta + h) = \sigma(I)$ is the ideal of A^T.

(3) *There is a $k[u]$-algebra isomorphism*
$$\tilde{\sigma} : R[u, u^{-d}I] \xrightarrow{\sim} A$$
which is $(G \times T)$-equivariant and restricts to the isomorphism $\sigma : R \xrightarrow{\sim} A^T$. Here, we consider $R[u, u^{-1}I]$ as the $(G \times T)$-subalgebra of $R[u, u^{-1}]$.

When the G-action on Y is trivial, we obtain by Lemma 4.4 that the kernel of a sequence Δ with a slice system is a polynomial ring with semi-invariant coordinates with respect to the T-action if the associated embedding φ_Δ is rectifiable. Forgetting the G-action, it is well-known that $\varphi_\Delta : V \to \tilde{V}$ is rectifiable in the cases (1) $m = 1$ and $n = 2$, (2) $m = n$, (3) $n \geq 2m + 2$. Hence in these three cases, A is a polynomial ring and u is a coordinate variable of A. In the case (1), it follows that $A = k[u, \tilde{v}, y]$ where $\tilde{v}$ (resp. y) is of degree 0 (resp. d) with respect to the T-action.

Corollary 4.5. *Suppose that $k = \mathbb{C}$ and $G = \mathbb{C}^*$. If $m = 1$ and $n = 2$, and a semi-invariant locally nilpotent derivation δ on $B = \mathbb{C}[u, v, x_1, x_2]$ of a form (1) has a slice, then A is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action.*

Proof. Let $X = \operatorname{Spec} A$. It follows from Lemma 4.1 (2) that $X/\!/T = \operatorname{Spec} A^T \cong \mathbb{A}^2$. Hence $\dim X/\!/(G \times T) \leq 2$. If $\dim X/\!/(G \times T) = 2$, then the G-action on $X \cong \mathbb{A}^3$ is trivial, and we are done. If $X/\!/(G \times T)$ is one-dimensional, the $(G \times T)$-action on X is linearizable by [8]. If $X/\!/(G \times T)$ is a point, then the $(G \times T)$-action on X is linearizable by [2]. □

We show that there is a bijective correspondence between the G-equivalence classes of G-embeddings and the G-equivalence classes of sequences with slice systems.

Lemma 4.6. (cf. [9, Lemma 3.6]) *Let $\Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)})$ and $\Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)})$ be two sequences having slice systems. Let φ_1 and φ_2 be the G-embeddings associated to Δ_1 and Δ_2, respectively. Then φ_1 and φ_2 are G-equivalent if and only if Δ_1 and Δ_2 are G-equivalent.*

Proof. Suppose that Δ_1 and Δ_2 are G-equivalent. Then there is a $(G\times T)$-equivariant S-automorphism ψ of B such that $\delta_i^{(2)}\circ\psi=\psi\circ\delta_i^{(1)}$ for every i. Let $A_{(1)}=\cap_{i=1}^m B^{\delta_i^{(1)}}$ and $A_{(2)}=\cap_{i=1}^m B^{\delta_i^{(2)}}$. Then ψ induces a G-equivariant isomorphism $\psi|_{A_{(1)}^T}:A_{(1)}^T\xrightarrow{\sim}A_{(2)}^T$. Let $u^d\theta_j^{(l)}+h_j^{(l)}$ $(1\le j\le n)$ be the element of $A_{(l)}^T$ defined as in Lemma 4.1 (1) with respect to Δ_l for $l=1,2$. Then the G-equivariant isomorphism $\sigma_l:R\xrightarrow{\sim}A_{(l)}^T$ is defined by $\sigma_l(X_j)=u^d\theta_j^{(l)}+h_j^{(l)}$ for $1\le j\le n$. The isomorphism $\psi|_{A_{(1)}^T}$ induces a G-equivariant automorphism $\gamma:R\to R$ such that $\gamma\circ\sigma_1^{-1}=\sigma_2^{-1}\circ\psi|_{A_{(1)}^T}$. Let $\Phi_l:R\to S$ be the G-equivariant surjection associated to Δ_l. Then noting that $\psi(u)=cu$ for $c\in k^*$, one obtains that for every j,

$$(\Phi_2\circ\gamma\circ\sigma_1^{-1})(u^d\theta_j^{(1)}+h_j^{(1)})=h_j^{(1)}=(\Phi_1\circ\sigma_1^{-1})(u^d\theta_j^{(1)}+h_j^{(1)})$$

(cf. the proof of [9, Lemma 3.6]). Hence it follows that $\Phi_1=\Phi_2\circ\gamma$, and φ_1 and φ_2 are G-equivalent.

Conversely, suppose that φ_1 and φ_2 are G-equivalent, i.e., $\Phi_1=\Phi_2\circ\gamma$ for a G-equivariant automorphism γ of R. Then it follows that $I_2=\gamma(I_1)$ where $I_l\subset R$ is the kernel of Φ_l. Hence γ extends to a $k[u]$-isomorphism $\tilde\gamma:R[u,u^{-d}I_1]\xrightarrow{\sim}R[u,u^{-d}I_2]$ which is $(G\times T)$-equivariant. By Lemma 4.4 (3), $\tilde\gamma$ induces a $k[u]$-isomorphism $\psi:A_{(1)}\xrightarrow{\sim}A_{(2)}$ which is $(G\times T)$-equivariant and satisfies $\sigma_2\circ\gamma=\psi\circ\sigma_1$. Let H_i $(1\le i\le m)$ be the element of R of weight α_i such that $\Phi_1(H_i)=v_i$, and let $w_i^{(1)}\in B$ $(1\le i\le m)$ be an element of degree (α_i,d) defined by $v_i-u^dw_i^{(1)}=\sigma_1(H_i)$. Then since $(w_1^{(1)},\ldots,w_m^{(1)})$ is the semi-invariant slice system of Δ_1, it follows that $B=A_{(1)}[w_1^{(1)},\cdots,w_m^{(1)}]$. We extend the $(G\times T)$-equivariant $k[u]$-isomorphism $\psi:A_{(1)}\to A_{(2)}$ to a $(G\times T)$-equivariant algebra homomorphism $\psi:B\to B$ by defining $\psi(w_i^{(1)})$ for $1\le i\le m$ by

$$u^d\psi(w_i^{(1)})=v_i-(\psi\circ\sigma_1)(H_i).$$

Note that $v_i-(\psi\circ\sigma_1)(H_i)\in u^dB$. In fact, since

$$\begin{aligned}(\psi\circ\sigma_1)(H_i) &= (\sigma_2\circ\gamma)(H_i)\\ &= (\gamma(H_i))(u^d\theta_1^{(2)}+h_1^{(2)},\ldots,u^d\theta_n^{(2)}+h_n^{(2)})\end{aligned}$$

and

$$(\gamma(H_i))(h_1^{(2)},\ldots,h_n^{(2)})=\Phi_2(\gamma(H_i))=\Phi_1(H_i)=v_i,$$

it follows that $v_i-(\psi\circ\sigma_1)(H_i)\in u^dB$ and $\psi(w_i^{(1)})\in B$ is defined. Let $w_i^{(2)}=\psi(w_i^{(1)})$ for $1\le i\le m$. Then by the definition of $\psi(w_i^{(1)})$, it is easily checked that $(w_1^{(2)},\ldots,w_m^{(2)})$ is a semi-invariant slice system of

Δ_2. Hence $B = A_{(2)}[w_1^{(2)}, \cdots, w_m^{(2)}]$. It follows that ψ is a $(G \times T)$-equivariant automorphism of B which satisfies $\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)}$ for every i. Furthermore, ψ is a S-automorphism of B since it follows from $v_i = u^d w_i^{(1)} + \sigma_1(H_i)$ for $1 \leq i \leq m$ that

$$\begin{aligned} \psi(v_i) &= u^d \psi(w_i^{(1)}) + (\psi \circ \sigma_1)(H_i) \\ &= v_i - (\psi \circ \sigma_1)(H_i) + (\psi \circ \sigma_1)(H_i) \\ &= v_i. \end{aligned}$$

□

By Proposition 4.3 and Lemma 4.6, there is a bijective correspondence between the G-equivalence classes of G-embeddings $V \to \tilde{V}$ and the G-equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ with slice systems.

For two sequences $\Delta_1 = (\delta_1^{(1)}, \ldots, \delta_m^{(1)})$ and $\Delta_2 = (\delta_1^{(2)}, \ldots, \delta_m^{(2)})$, we define that Δ_1 and Δ_2 are weakly G-equivalent iff there is a $(G \times T)$-equivariant automorphism ψ of B, not necessarily an S-automorphism, such that $\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)}$ for every i. We define also that two G-equivariant embeddings φ_1 and φ_2 of V into $\tilde{V}$ are weakly G-equivalent iff there is a G-equivariant automorphism γ of R such that $I_2 = \gamma(I_1)$ where I_l is the kernel of the surjection $\Phi_l : R \to S$ associated with φ_l for $l = 1, 2$. Then there is a bijective correspondence between the weakly G-equivalence classes of G-embeddings $V \to \tilde{V}$ and the weakly G-equivalence classes of sequences $\Delta = (\delta_1, \ldots, \delta_m)$ with slice systems. In fact, Lemma 4.6 holds true when replacing "G-equivalent" by "weakly G-equivalent". The difference between "G-equivalent" and "weakly G-equivalent" is whether one admits the exchange of a system of semi-invariant coordinates of $V = \operatorname{Spec} S$ or not. Note that a G-embedding $\varphi : V \to \tilde{V}$ is G-rectifiable iff φ is weakly G-equivalent to the standard G-embedding $V \to V \oplus V' = \tilde{V}$.

Lemma 4.7. *Let $\Delta_l = (\delta_1^{(l)}, \ldots, \delta_m^{(l)})$ and φ_l be the same as in Lemma 4.6 for $l = 1, 2$. Let $A_{(l)} = \cap_{i=1}^m B^{\delta_i^{(l)}}$ for $l = 1, 2$. Then the following are equivalent.*

1. *φ_1 and φ_2 are weakly G-equivalent.*
2. *Δ_1 and Δ_2 are weakly G-equivalent.*
3. *$A_{(1)}$ is $(G \times T)$-equivariantly isomorphic to $A_{(2)}$.*

Proof. The equivalence between (1) and (2) can be proved as in the proof of [9, Lemma 3.7]. It is obvious that (3) follows from (2). We show that (3) implies (2). Suppose that (3) holds. Then there is a $(G \times T)$-equivariant isomorphism $\psi : A_{(1)} \xrightarrow{\sim} A_{(2)}$. The isomorphism ψ extends to a $(G \times T)$-automorphism ψ of B such that $\delta_i^{(2)} \circ \psi = \psi \circ \delta_i^{(1)}$ for every i. In fact, let $(w_1^{(1)}, \ldots, w_m^{(1)})$ (resp. $(w_1^{(2)}, \ldots, w_m^{(2)})$) be any semi-invariant slice system of Δ_1 (resp. Δ_2). Then $B = A_{(1)}[w_1^{(1)}, \ldots, w_m^{(1)}] = A_{(2)}[w_1^{(2)}, \ldots, w_m^{(2)}]$. Define an algebra homomorphism $\psi : B \to B$ by extending $\psi : A_{(1)} \to A_{(2)}$ by $\psi(w_i^{(1)}) = w_i^{(2)}$. Then $\psi : B \to B$ is the required automorphism and Δ_1 and Δ_2 are weakly G-equivalent. □

5. The $(G \times T)$-action on X and G-embeddings

In this section, we continue the notation in the previous section and observe the G-embedding φ_Δ associated to a sequence Δ on B and the $(G \times T)$-action on $X = \operatorname{Spec} A$.

Let $\Delta = (\delta_1, \ldots, \delta_m)$ be a sequence with a slice system on $B = k[u, v_1, \ldots, v_m, x_1, \ldots, x_n]$. Then as observed in the previous section, it follows that $m \le n$. Let $(w_1, \ldots, w_m)$ be a semi-invariant slice system of Δ. Then it follows that $B = A[w_1, \cdots, w_m]$, i.e.,

$$X \times V_{-d} \cong Y = L \oplus V_0 \oplus V_{-d} \oplus V'_{-d} \tag{3}$$

where $X = \operatorname{Spec} A$. If X is isomorphic to a $(G \times T)$-representation, then it follows from (3) that X is necessarily isomorphic to the $(G\times T)$-representation

$$W := L \oplus V_0 \oplus V'_{-d}.$$

If the associated G-embedding φ_Δ is G-rectifiable, then it follows from Lemma 4.4 that $X \cong W$, namely, the equivariant cancellation holds. By Lemma 4.7, it follows that the associated G-embedding φ_Δ is G-rectifiable if and only if $X \cong W$.

Theorem 5.1. (cf. [9, Theorem 5.1]) *Suppose that a sequence Δ has a slice system. Then X is isomorphic to W if and only if the associated G-embedding φ_Δ is G-rectifiable.*

As observed in section 2, the m-dimensional torus T' acts on $Y = \operatorname{Spec} B$ where $T' = T_1 \times \cdots \times T_m$ with $T_i = G_m$ for $1 \le i \le m$. The T_i-action on Y corresponds to the $\mathbb{Z}$-grading on $B = A[w_1, \ldots, w_m]$ such that $\deg w_i = 1$ and $\deg a = 0$ for $a \in B^{\delta_i} - \{0\}$. Since the T'-action on Y commutes with the $(G\times T)$-action, an $(m+1)$-dimensional torus $(T \times T')$ acts on Y. Note that $B^{T_i} = B^{\delta_i}$ and $B^{T'} = A$. Hence $X = Y//T'$. The algebraic quotient $Y//T' = \operatorname{Spec} B^{T'}$ is $(G \times T)$-equivariantly isomorphic to the fixed-point locus $Y^{T'} = \operatorname{Spec} B/(w_1, \ldots, w_m)$.

Proposition 5.2. *The $(G \times T \times T')$-action on Y is linearizable if and only if $X \cong W$.*

Proof. If the $(G \times T \times T')$-action on Y is linearizable, then $X = Y//T'$ is isomorphic to a $(G \times T)$-representation, which must be W. Conversely, suppose that $X \cong W$. Then since the $(G \times T \times T')$-variety Y is a product of a $(G \times T)$-variety X with a trivial T'-action and a $(G \times T \times T')$-representation, the $(G \times T \times T')$-action on Y is linearizable. □

By Theorem 5.1 and Proposition 5.2, we have the following.

Corollary 5.3. *The $(G \times T \times T')$-action on Y is linearizable if and only if φ_Δ is G-rectifiable.*

It is known that the $(T \times T')$-action on Y is linearizable iff φ_Δ is rectifiable (cf. [9, Lemma 4.1]).

In the following, we assume that k is algebraically closed. Note that W is isomorphic to $L \oplus \tilde{V}$ as a G-representation.

Proposition 5.4. *Let Δ be a sequence on B having a slice system. Suppose that the G-action on Y is fix-pointed and $V^G = \{0\}$. Then $X \cong L \oplus \tilde{V}$ as a G-representation.*

Proof. By Lemma 2.2, it follows that $X \cong X//G \times W'$ for a G-representation W'. Note that $X//G \cong X^G$ since the G-action on X is fix-pointed as well. Since $V^G = \{0\}$, it follows from (3) that $X^G \cong L \oplus (V'_{-d})^G$, hence $X//G$ is an affine space. Therefore X is isomorphic to a G-representation, hence $X \cong L \oplus \tilde{V}$. □

Remark. Under the assumption in Proposition 5.4, the G-action on $X_i = \operatorname{Spec} A_i$ is fix-pointed where $A_i = B^{\delta_i}$. Hence it follows that every X_i is isomorphic to a G-representation.

Suppose that G is an r-dimensional torus $(k^*)^r$. Then B is $\mathbb{Z}^r$-graded. If every component of $\deg v_i$ is positive and every component of $\deg x_j$ is non-negative, then the G-action on Y is fix-pointed and $V^G = \{0\}$. By Proposition 5.4, it follows that $X \cong L \oplus \tilde{V}$.

We obtain the following result on G-embeddings.

Theorem 5.5. *Let G be a complex torus. Let V and $\tilde{V}$ be complex G-representations and let $\dim \tilde{V}//G \leq 1$. Then every G-embedding $V \to \tilde{V}$ is G-rectifiable.*

Proof. Let $\varphi : V \to \tilde{V}$ be a G-embedding. By Theorem 5.1 and Proposition 4.3, it suffices to show that $X = \operatorname{Spec} A$ is isomorphic to a $(G \times T)$-representation where A is the kernel of the sequence Δ_φ associated to φ. Note that X is smooth and acyclic by (3). Recall that $X//T = \mathrm{Spec}\, A^T$ is isomorphic to the G-representation $\tilde{V}$. Hence the algebraic quotient $X//(G \times T) \cong \tilde{V}//G$ is of dimension ≤ 1. By the results of [2] and [8], X is isomorphic to the $(G \times T)$-representation. Hence the assertion follows. □

Let $p_L : X \to L$ be the morphism corresponding to the inclusion $k[u] \hookrightarrow A$. Then p_L is G-equivariant. Since the derivation δ_i on B_u has a slice v_i/u^d, it follows from Lemma 2.4 that $p_L^{-1}(U)$ is G-equivariantly isomorphic to $U \times \tilde{V}$ where $U = \mathrm{Spec}\, k[u]_u$.

Lemma 5.6. (cf. [9, Lemma 4.2]) *The morphism $p_L : X \to L$ is flat and every closed fiber of p_L is isomorphic to $\tilde{V}$.*

Proof. Let $\tilde{p}_L : Y \to L$ be the projection. Then the T'-action on L is trivial and $\tilde{p}_L$ is $(G \times T \times T')$-equivariant. It holds that $\tilde{p}_L = p_L \circ \pi$ where $\pi : Y \to Y//T' = X$ is the quotient. For a closed point $c \in L \cong \mathbb{A}^1$, $\tilde{p}_L^{-1}(c)$ is T'-stable and $p_L^{-1}(c) = \pi(\tilde{p}_L^{-1}(c)) = \tilde{p}_L^{-1}(c)//T'$ since π is surjective. From the above remark, $p_L^{-1}(c) \cong \tilde{V}$ for a closed point $c \in U$. We show that $p_L^{-1}(0) \cong V_0 \oplus V'_{-d}$. The T_i-action on $\tilde{p}_L^{-1}(0) = \mathrm{Spec}\, \bar{B}$ is induced by $\bar{\delta}_i$ and its slice $\bar{w}_i$ where $\bar{B} = B/(u)$, $\bar{\delta}_i$ is the locally nilpotent derivation on $\bar{B}$ induced by δ_i, and $\bar{w}_i$ is the residue class of w_i in $\bar{B}$. Since $\bar{B} = \bar{A}[\bar{w}_1, \ldots, \bar{w}_m]$ where $\bar{A} = \cap_{i=1}^m \bar{B}^{\bar{\delta}_i}$, it follows that $p_L^{-1}(0) = \tilde{p}_L^{-1}(0)//T' = \mathrm{Spec}\, \bar{A}$. Each derivation $\bar{\delta}_i$ on $\bar{B} \cong k[v_1, \ldots, v_m, x_1, \ldots, x_n]$ is a semi-invariant $k[v_1, \ldots, v_m]$-derivation of weight $(-\alpha_i, -d)$ having a slice and satisfies $\bar{\delta}_i(x_j) \in k[v_1, \ldots, v_m]$ for $1 \leq j \leq n$. By Lemma 2.3, it follows that $\bar{B}^{\bar{\delta}_1} = k[v_1, \ldots, v_m, \xi_{1,2}, \ldots, \xi_{1,n}]$ where $\xi_{1,j}$ is of weight (β_j, d) and linear in $x_1, \ldots, x_n$ over $k[v_1, \ldots, v_m]$. Note that the locally nilpotent derivation $\bar{\delta}_2$ on $\bar{B}^{\bar{\delta}_1}$ has a slice and $\bar{\delta}_2(\xi_{1,j}) \in k[v_1, \ldots, v_m]$ for $2 \leq j \leq n$. By applying Lemma 2.3 to $\bar{\delta}_2$ on $\bar{B}^{\bar{\delta}_1}$, we have $\bar{B}^{\bar{\delta}_1} \cap \bar{B}^{\bar{\delta}_2} = k[v_1, \ldots, v_m, \xi_{2,3}, \ldots, \xi_{2,n}]$ where $\xi_{2,j}$ is of weight (β_j, d) and linear in $x_1, \ldots, x_n$ over $k[v_1, \ldots, v_m]$. Hence by applying Lemma 2.3 subsequently to $\bar{\delta}_i$ on $\cap_{l=1}^{i-1} \bar{B}^{\bar{\delta}_l}$ for $2 \leq i \leq m$, we obtain that $\bar{A}$ is a polynomial ring over $k[v_1, \ldots, v_m]$ in semi-invariant $n - m$ variables and $p_L^{-1}(0) \cong V_0 \oplus V'_{-d}$. Since every closed fiber $p_L^{-1}(c)$ is isomorphic to $\mathbb{A}^n$, it follows from [10, 6], [7] that p_L is flat. □

If $p_L : X \to L$ is an algebraic G-vector bundle, then X is isomorphic to a G-representation. In fact, for a reductive group G, it is well-known that every algebraic G-vector bundle over a G-representation L of dimension one is trivial, i.e., isomorphic to a product bundle $L \times F$

for a G-representation F. Note that u is a coordinate variable of a polynomial ring A when $p_L : X \to L$ is an algebraic G-vector bundle.

Proposition 5.7. *Let Δ be a sequence having a slice system. Then $p_L : X \to L$ is an algebraic $(G \times T)$-vector bundle if and only if the associated G-embedding φ_Δ is G-rectifiable.*

Proof. Suppose that $p_L : X \to L$ is an algebraic $(G \times T)$-vector bundle. Since the fiber $p_L^{-1}(0)$ is isomorphic to $V_0 \oplus V'_{-d}$ (cf. the proof of Lemma 5.6), it follows that $X \cong L \oplus V_0 \oplus V'_{-d} = W$. Hence by Theorem 5.1, φ_Δ is G-rectifiable. Conversely, suppose that φ_Δ is G-rectifiable. Then by Lemma 4.4, A is a polynomial ring with semi-invariant coordinates with respect to the $(G \times T)$-action and u is a semi-invariant coordinate variable of A. Hence $p_L : X \to L$ is a trivial $(G \times T)$-vector bundle. □

Finally, we give three examples and discuss problems related to them.

Example 5.1. Let $V = \operatorname{Spec} \mathbb{R}[v]$ and $\tilde{V} = \operatorname{Spec} \mathbb{R}[x_1, x_2, x_3]$ be $\mathbb{Z}/2\mathbb{Z}$-representations with weight 1 and $(1, 0, 1)$, respectively, namely, with the linear $\mathbb{Z}/2\mathbb{Z}$-actions

$$\tau \cdot v = -v, \qquad \tau \cdot (x_1, x_2, x_3) = (-x_1, x_2, -x_3)$$

for a generator $\tau \in \mathbb{Z}/2\mathbb{Z}$. Let $\varphi : V \to \tilde{V}$ be an embedding associated with the surjection $\Phi : \mathbb{R}[x_1, x_2, x_3] \to \mathbb{R}[v]$ defined by $\Phi(x_1) = v^3$, $\Phi(x_2) = v^4$, $\Phi(x_3) = v^5 + v$ (cf. [3]). Then φ is $\mathbb{Z}/2\mathbb{Z}$-equivariant. The locally nilpotent derivation δ on $B = \mathbb{R}[u, v, x_1, x_2, x_3]$ associated to φ is

$$\delta = u^d \partial_v + 3v^2 \partial_{x_1} + 4v^3 \partial_{x_2} + (5v^4 + 1)\partial_{x_3}$$

which is semi-invariant of weight $(1, -d)$. It is known that φ is rectifiable by [3]. In fact, the $\mathbb{Z}/2\mathbb{Z}$-equivariant automorphism Ψ of $\mathbb{R}[x_1, x_2, x_3]$ given by

$$\begin{aligned}
\Psi(x_1) &= x_1^3 x_2 + 2x_1^3 + x_3 - x_1 x_3^2 \\
\Psi(x_2) &= -2 - 5x_3^4 + 6x_1 x_3^5 - 2x_1^2 x_3^6 \\
&\quad + (1 + 2x_1 x_3 + 4x_1^2 x_3^2 - 12x_1^3 x_3^3 + 6x_1^4 x_3^4)(x_2 + 2) \\
&\quad + (x_1^4 + 6x_1^5 x_3 - 6x_1^6 x_3^2)(x_2 + 2)^2 + 2x_1^8 (x_2 + 2)^3 \\
\Psi(x_3) &= x_1 - (x_1^3 x_2 + 2x_1^3 + x_3 - x_1 x_3^2)^3
\end{aligned}$$

rectifies φ into the standard $\mathbb{Z}/2\mathbb{Z}$-embedding. Hence the kernel $A = B^\delta$ is $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-equivariantly isomorphic to a polynomial ring $\mathbb{R}[u, \tilde{v}, y_2, y_3]$ where the weight of u, $\tilde{v}$, y_2 and y_3 is $(0, -1)$, $(1, 0)$, $(0, d)$ and $(1, d)$, respectively. Therefore $X = \operatorname{Spec} A$ is a $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-representation $W = \operatorname{Spec} \mathbb{R}[u, \tilde{v}, y_2, y_3]$.

Example 5.2. Let $V^* = \operatorname{Spec} \mathbb{C}[v]$ and $\tilde{V}^* = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3]$ be the $\mathbb{Z}/4\mathbb{Z}$-representations on which the $\mathbb{Z}/4\mathbb{Z}$-actions are given by

$$\lambda \cdot v = \zeta v, \qquad \lambda \cdot (x_1, x_2, x_3) = (\zeta^3 x_1, x_2, \zeta x_3)$$

where λ is a generator of $\mathbb{Z}/4\mathbb{Z}$ and ζ is the 4-th primitive root of unity. As a representation of the subgroup $\mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}/4\mathbb{Z}$, $V^* = V \otimes_{\mathbb{R}} \mathbb{C} =: V_{\mathbb{C}}$ and $\tilde{V}^* = \tilde{V} \otimes_{\mathbb{R}} \mathbb{C} =: \tilde{V}_{\mathbb{C}}$. The surjection $\Phi : \mathbb{R}[x_1, x_2, x_3] \to \mathbb{R}[v]$ extends to a surjection $\mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[v]$ and the extended surjection defines a $\mathbb{Z}/4\mathbb{Z}$-embedding $\varphi^* : V^* \to \tilde{V}^*$ which restricts to a $\mathbb{Z}/2\mathbb{Z}$-embedding $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to \tilde{V}_{\mathbb{C}}$. The locally nilpotent derivation $\delta_{\mathbb{C}}$ associated to φ^* is of the same form as δ and semi-invariant of weight $(-1, -d)$. Let $A_{\mathbb{C}} = B_{\mathbb{C}}^{\delta_{\mathbb{C}}}$ where $B_{\mathbb{C}} = \mathbb{C}[u, v, x_1, x_2, x_3]$. The automorphism Ψ induces a $\mathbb{Z}/2\mathbb{Z}$-equivariant automorphism $\Psi_{\mathbb{C}}$ of $\mathbb{C}[x_1, x_2, x_3]$, which rectifies $\varphi_{\mathbb{C}}$. Hence $X_{\mathbb{C}} = \operatorname{Spec} A_{\mathbb{C}}$ is isomorphic to the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^*)$-representation $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$. The automorphism $\Psi_{\mathbb{C}}$ is not $\mathbb{Z}/4\mathbb{Z}$-equivariant and we do not know whether the $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{C}^*)$-action on $X \cong \mathbb{A}^4_{\mathbb{C}}$ is linearizable or not.

Example 5.3. Let V and $\tilde{V}$ be as in Example 5.1 and let $\varphi : V \to \tilde{V}$ be a $\mathbb{Z}/2\mathbb{Z}$-embedding associated with the surjection Φ defined by $\Phi(x_1) = v^3 - 3v$, $\Phi(x_2) = v^4 - 4v^2$, $\Phi(x_3) = v^5 - 10v$. The locally nilpotent derivation on B associated to φ is

$$\delta = u^d \partial_v + (3v^2 - 3)\partial_{x_1} + (4v^3 - 8v)\partial_{x_2} + (5v^4 - 10)\partial_{x_3}$$

which is semi-invariant of weight $(1, -d)$. It is known that φ is non-rectifiable by [11] and [1]. Hence X is not isomorphic to the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-representation W. It follows from [1] and [9] that X is not $\mathbb{R}^*$-equivariantly isomorphic to W, neither. However, it is not known whether X is isomorphic to $\mathbb{A}^4_{\mathbb{R}}$ or not if we forget the action. If $X \cong \mathbb{A}^4_{\mathbb{R}}$, then it follows that the $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^*)$-action on $X \cong \mathbb{A}^4_{\mathbb{R}}$ is non-linearizable. By Corollary 5.3, the $(\mathbb{Z}/2\mathbb{Z} \times (\mathbb{R}^*)^2)$-action on $Y \cong \mathbb{A}^5_{\mathbb{R}}$ is non-linearizable (cf. [1]). It is unknown neither whether there exists a non-linearizable action of a finite subgroup of $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{R}^*)^2$ on Y. The surjection Φ induces a surjection $\mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}[v]$ which defines a $\mathbb{Z}/2\mathbb{Z}$-embedding $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to \tilde{V}_{\mathbb{C}}$. It remains open whether $\varphi_{\mathbb{C}}$ is rectifiable or not.

References

[1] T. Asanuma, Non-linearizable algebraic k^*-actions on affine spaces, Invent. Math. **138** (1999), 281–306.

[2] H. Bass and W. Haboush, Linearizing certain reductive group actions, Trans. Amer. Math. Soc. **292** (1985), 463–482.

[3] P. C. Craighero, About Abhyankar's conjectures on space lines, Rend. Sem. Mat. Univ. Padova, vol. 74 (1985), 115–122.

[4] A. van den Essen, Polynomial automorphisms, Progress in Mathematics vol. 190, Birkhäuser, Basel-Boston-Berlin, 2000.

[5] A. van den Essen and P. van Rossum, Triangular derivations related to problems on affine n-space, Proc. Amer. Math. Soc. **130** (2001), 1311–1322.

[6] G. Freudenburg, Algebraic theory of locally nilpotent derivations, Encyclopaedia of Mathematical Sciences vol. 136, Springer, Berlin-Heidelberg-New York, 2006.

[7] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique, Publ. Math. I.H.E.S. **24** (1965).

[8] H. Kraft and G. Schwarz, Reductive group actions with one-dimensional quotient, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 1–97.

[9] K. Masuda, Homogeneous locally nilpotent derivations having slices and embeddings of affine spaces, J. Algebra **321** (2009), 1719–1733.

[10] M. Miyanishi, Algebraic Geometry, Translations of Math. Monographs vol. 136, Amer. Math. Soc. 1994.

[11] A. Shastri, Polynomial representations of knots, Tohoku Math. J. **44** (1992), 11–17.

Department of Mathematical Sciences, Faculty of Science and Technology, Kwansei Gakuin University, 2-1 Gakuen, Sanda 669-1337, Japan

E-mail address: `kayo@kwansei.ac.jp`