CERTAIN MODULI OF ALGEBRAIC G-VECTOR BUNDLES OVER AFFINE G-VARIETIES

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Abstract. Let G be a reductive complex algebraic group and P a complex G-module with algebraic quotient of dimension ≥ 1. We construct a map from a certain moduli space of algebraic G-vector bundles over P to a $\mathbb{C}$-module possibly of infinite dimension, which is an isomorphism under some conditions. We also show non-triviality of moduli of algebraic G-vector bundles over a G-stable affine hypersurface of some type. In particular, we show that the moduli space of algebraic G-vector bundles over a G-stable affine quadric with fixpoints and one-dimensional quotient contains $\mathbb{C}^p$.

Introduction and results

Let G be a reductive algebraic group defined over the ground field $\mathbb{C}$ of complex numbers. One of the most important problems in the theory of algebraic group action is to understand algebraic G-actions on affine space $\mathbb{A}^n$. The following problem is fundamental;

Linearization Problem

Is every action of G on $\mathbb{A}^n$ linearizable, i.e., conjugate to a linear action under polynomial automorphisms of $\mathbb{A}^n$?

In 1989, Schwarz [23] presented the first examples of *non-linearizable* actions on affine space. In fact, he first showed that there exist *non-trivial* algebraic G-vector bundles over G-modules, and the non-lineari zable actions appear on the total spaces of non-trivial algebraic G-vector bundles he found. An algebraic G-vector bundle E over an affine G-variety X is an algebraic vector bundle $p : E \to X$ together with a G-action on E such that p is G-equivariant and the action on the fibers is linear. By definition, every fiber over the fixpoint locus X^G is a G-module. An algebraic G-vector bundle is called trivial if it is isomorphic to a G-vector bundle of the form $X \times Q \to X$ for a G-module Q. When the base space is a G-module, if forgetting the G-action, the total space E is an affine space by the affirmative solution to the Serre Conjecture by Quillen [22] and Suslin [25]. So, the G-action

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on the total space of a non-trivial G-vector bundle over a G-module is a candidate for a non-linearizable action on affine space. In fact, there are some criteria for the G-action on E being non-linearizable ([1],[7], [18]). So far, all known examples of non-linearizable action are obtained from non-trivial algebraic G-vector bundles. For an abelian G, at this point, there are no counterexamples to the Linearization Problem; for, by Masuda-Moser-Petrie [19], every G-vector bundle over a G-module is trivial when G is abelian. The key point of their proof is to show that one can reduce triviality of a G-vector bundle over a G-module P to triviality of a vector bundle over the algebraic quotient space $P//G$ (=the spectrum of the ring of G-invariant polynomials on P). Since G is abelian, $P//G$ is a normal affine toric variety, and triviality of a vector bundle over a normal affine toric variety was obtained by Gubeladze [5]. We refer to Kraft [10] for recent topics in affine algebraic geometry and algebraic group action related to the Linearization Problem.

In this article, we study algebraic G-vector bundles over affine G-varieties X, especially in the case that X is a G-module. Throughout this article, we assume that X is irreducible and smooth and that X^G is non-empty. We denote by $\mathrm{VEC}_G(X, Q)$ the set of equivariant isomorphism classes of algebraic G-vector bundles over X such that every fiber over X^G is isomorphic to a G-module Q. The isomorphism class of a G-vector bundle $E \to X$ is denoted by $[E]$. Suppose that the base space is a G-module P. In this case, we have some information on $\mathrm{VEC}_G(P, Q)$ ([1], [2], [23], [11], [6], [18], [20]). By Bass-Haboush [1], every G-vector bundle over P is stably trivial, i.e., there exists a G-module S such that a Whitney sum $E \oplus (P \times S)$ is trivial. For an abelian G, $\mathrm{VEC}_G(P, Q)$ is trivial, i.e., a trivial set consisting of the trivial class $[P \times Q]$ by Masuda-Moser-Petrie [19]. For a non-abelian G, if the dimension of $P//G$ is at most one, $\mathrm{VEC}_G(P, Q)$ is well-understood. When $\dim P//G = 0$, $\mathrm{VEC}_G(P, Q)$ is trivial ([2], [12]). When $\dim P//G = 1$, however, $\mathrm{VEC}_G(P, Q)$ can be non-trivial. Schwarz ([23], cf. Kraft-Schwarz [11]) showed that $\mathrm{VEC}_G(P, Q)$ is isomorphic to an additive group $\mathbb{C}^p$ for a nonnegative integer p, and the non-trivial G-vector bundles found by Schwarz led to the first examples of non-linearizable actions on affine space, as is already mensioned above. The result of Schwarz extends to the case where the base space is a (not necessarily irreducible) G-stable affine cone X with one-dimensional quotient, namely, it holds that $\mathrm{VEC}_G(X, Q) \cong \mathbb{C}^p$ for some p ([21],[15]). However, when $\dim P//G \geq 2$, $\mathrm{VEC}_G(P, Q)$ is not finite-dimensional any more. In fact, $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q)$ for a G-module P with one-dimensional quotient and a trivial G-module $\mathbb{C}^m$ is isomorphic to the p times direct product of a polynomial ring $\mathbb{C}[y_1, \cdots, y_m]$

where p is a nonnegative integer such that $\mathrm{VEC}_G(P,Q) \cong \mathbb{C}^p$ [16]. Furthermore, Mederer [21] presented examples of $\mathrm{VEC}_G(P,Q)$ which contains an uncountably-infinite dimensional space for a finite group G. Using Mederer's result, it is shown that $\mathrm{VEC}_G(P,Q)$ can contain an uncountably-infinite dimensional space also for a connected group G [17]. However, $\mathrm{VEC}_G(X,Q)$ are not yet classified even when X is a G-module P with $\dim P/\!/G \geq 2$ except some special cases ([6], cf. [20]) and the cases mensioned above.

We denote by $\mathcal{O}(X)$ the $\mathbb{C}$-algebra of regular functions on X and by $\mathcal{O}(X)^G$ the subalgebra of G-invariants of $\mathcal{O}(X)$. By the finiteness theorem of Hilbert, $\mathcal{O}(X)^G$ is finitely generated and the algebraic quotient space $X/\!/G$ is defined to be $\mathrm{Spec}\,\mathcal{O}(X)^G$. Let $\pi_X : X \to X/\!/G$ be the algebraic quotient map, that is, the morphism induced by the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$. Since X is irreducible, $X/\!/G$ is an irreducible affine variety (cf. [8]). By Luna's slice theorem [12], there is a finite stratification of $X/\!/G = \cup_i V_i$ into locally closed subvarieties V_i such that $\pi_X|_{\pi_X^{-1}(V_i)} : \pi_X^{-1}(V_i) \to V_i$ is a G-fiber bundle (in the étale topology) and the isotropy groups of closed orbits in $\pi_X^{-1}(V_i)$ are all conjugate to a fixed reductive subgroup H_i. The unique open dense stratum of $X/\!/G$, which we denote by U, is called the principal stratum and the corresponding isotropy group, which we denote by H, is called a principal isotropy group. The principal isotropy group H is the minimal group among H_i up to conjugation. Suppose that $\dim X/\!/G \geq 1$. We denote by $\mathrm{VEC}_G(X,Q)_0$ the subset of $\mathrm{VEC}_G(X,Q)$ consisting of elements which are trivial over $\pi_X^{-1}(U)$ and $\pi_X^{-1}(V)$ with fiber Q where $V := X/\!/G - U$. Though we do not know how to compute $\mathrm{VEC}_G(X,Q)$, it is not difficult to analyse $\mathrm{VEC}_G(X,Q)_0$ since every $[E] \in \mathrm{VEC}_G(X,Q)_0$ is determined by a transition function with respect to two trivializations of E. In the case that X is a (not necessarily irreducible) G-stable affine cone with $\dim X/\!/G = 1$, in particular, a G-module with one-dimensional quotient, $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)$ and $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)_0$ coincide and we can compute $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)_0$ by analysing transition functions ([11], [16]). We assume that the ideal of V is principal; for, if $[E] \in \mathrm{VEC}_G(X,Q)$ is trivial over $\pi_X^{-1}(U)$ such that $\pi_X^{-1}(V)$ is of codimension ≥ 2, then E is trivial. Our first result is a classification of $\mathrm{VEC}_G(P,Q)_0$ for a G-module P with $\dim P/\!/G \geq 2$.

Theorem 1. *Let P be a G-module such that $\dim P/\!/G \geq 2$ and the ideal of the complement of the principal stratum in $P/\!/G$ is principal. Let Q be a G-module. Then there exists a map*

$$\Psi_{P,Q} : \mathrm{VEC}_G(P,Q)_0 \to C_P(Q).$$

Here $C_P(Q)$ is a $\mathbb{C}$-module possibly of infinite dimension (cf. 2.3). If Q is multiplicity free with respect to a principal isotropy group of P and if P has generically closed orbits, then $\Psi_{P,Q}$ is an isomorphism.

Here, a G-module Q is called "multiplicity free with respect to a reductive subgroup H" if every irreducible H-module appears in Q, viewed as an H-module, with multiplicity at most one, and we say "P has generically closed orbits" if every fiber of the quotient map π_P over the principal stratum consists of a closed orbit.

For any G-module P with one-dimensional quotient and any Q, $\Psi_{P\oplus\mathbb{C}^m,Q}$ in Theorem 1 is an isomorphism onto $C_{P\oplus\mathbb{C}^m}(Q) \cong (\mathbb{C}[y_1,\cdots,y_m])^p$, which coincides with the isomorphism obtained in [16].

Next, we investigate $\mathrm{VEC}_G(X,Q)_0$ for an affine quadric X. An affine quadric of dimension N is an affine hypersurface $X := \{(x_0,\cdots,x_N) \in \mathbb{A}^{N+1} \mid \sum_{i=0}^N x_i^2 = 1\}$. We suppose that G is connected and acts on an affine quadric X in such a way that the kernel of the action is finite. Suppose also that X^G is not empty and $\dim X/\!/G = 1$. Then by Doebeli ([3] [4]), X is G-isomorphic to an affine quadric $X_P := \{(x,v) \in P\oplus\mathbb{C} \mid u(x)+v^2 = 1\}$, where P is an orthogonal G-module with $P/\!/G \cong \mathbb{A}^1$ and $u(x) \in \mathcal{O}(P)^G$ is an invariant quadratic form generating $\mathcal{O}(P)^G$. The G-action on X_P is the one induced by the linear action on P. This time, however, the situation is rather different from that in the case of G-modules. The fixpoint locus X_P^G consists of two points $\{(O,\pm1)\}$ where O is the origin of P, whereas the fixpoint locus of a G-module is an affine space, hence connected. Though $X_P/\!/G$ is isomorphic to $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[v]$, V of $X_P/\!/G$ consists of two points $\{v=\pm1\}$, hence V of $X_P/\!/G$ is disconnected. For a G-module, V is connected since V is defined by invariant homogeneous polynomials. Thus we cannot apply methods in case of G-modules directly to a case of an affine quadric. While, note that X is viewed as a $G\times(\mathbb{Z}/2\mathbb{Z})$-variety, where $\mathbb{Z}/2\mathbb{Z}$ acts on $X \cong X_P \subset P\oplus\mathbb{C}$ via a (non-trivial) linear action on $\mathbb{C}$. Then $X/(\mathbb{Z}/2\mathbb{Z}) \cong P$ as a G-variety. It is easy to see that the quotient map $\pi_{\mathbb{Z}_2} : X \to X/(\mathbb{Z}/2\mathbb{Z}) \cong P$ induces an injection $\pi_{\mathbb{Z}_2}^* : \mathrm{VEC}_G(P,Q) \to \mathrm{VEC}_G(X,Q)$ (cf. [9]). Since $\mathrm{VEC}_G(P,Q) \cong \mathbb{C}^p$ by the result of Schwarz, $\mathrm{VEC}_G(X,Q)$ contains a space isomorphic to $\mathbb{C}^p$. We generalize this and obtain the following result.

Theorem 2. *Let P be a G-module with $\dim P/\!/G \geq 1$. For $f \in \mathcal{O}(P)^G$ and an integer $d \geq 2$, let $X_P(f,d)$ be a G-stable hypersurface $\{(x,v) \in P\oplus\mathbb{C} \mid f(x)+v^d = 1\}$. Then, the quotient map $\pi_{\mathbb{Z}_d} : X_P(f,d) \to X_P(f,d)/(\mathbb{Z}/d\mathbb{Z}) \cong P$ induces an injection for any G-module Q*

$$\pi_{\mathbb{Z}_d}^* : \mathrm{VEC}_G(P,Q) \to \mathrm{VEC}_G(X_P(f,d),Q).$$

Hence, if $\Psi_{P,Q}$ in Theorem 1 is a surjection onto a non-trivial $C_P(Q)$, then $\mathrm{VEC}_G(X_P(f,d),Q)$ is non-trivial, too.

This article consists of three parts. In section 1, we investigate $\mathrm{VEC}_G(X,Q)_0$ for an irreducible smooth affine G-variety X by analysing transition functions of G-vector bundles. We have in mind as an X a G-module. Our technique is based on the one established by Kraft-Schwarz [11]. Using the results obtained in section 1, we prove Theorem 1 in section 2. We compute $\mathrm{VEC}_G(P,Q)_0$ explicitly in examples. In section 3, we ingestigate $\mathrm{VEC}_G(X,Q)_0$ in the case where V is not connected, in particular, in the case where X is a G-stable affine hypersurface represented by an affine quadric with fixpoints and one-dimensional quotient.

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1. General results

Let G be a reductive algebraic group and X an irreducible smooth affine G-variety. We assume that the dimension of $Y := X//G$ is greater than 0 and the ideal of $V = Y - U$ is principal, where U is the principal stratum of Y. Let $f \in \mathcal{O}(Y) = \mathcal{O}(X)^G$ be a generator of the ideal of V. We assume also that X^G is non-empty, connected and X^H is irreducible where H is a principal isotropy group of X. The object we have in mind as an X is a G-module. We will investigate $\mathrm{VEC}_G(X,Q)_0$ for a G-module Q.

Lemma 1.1. *Let $[E] \in \mathrm{VEC}_G(X,Q)_0$. Then E is trivial over $X_h := \{x \in X \mid h(x) \neq 0\}$ where h is an element of $\mathcal{O}(Y)$ such that $h-1$ is contained in the ideal (f).*

Proof. Since $E|_{\pi_X^{-1}(V)}$ is, by the assumption, isomorphic to a trivial bundle, it follows from the Equivariant Nakayama Lemma [2] that the trivialization $E|_{\pi_X^{-1}(V)} \to \pi_X^{-1}(V) \times Q$ extends to a trivialization over a G-stable open neighborhood $\tilde{U}$ of $\pi_X^{-1}(V)$. Let $\tilde{V}$ be the complement of $\tilde{U}$ in X. Since $\tilde{V}$ is a G-stable closed set, $\pi_X(\tilde{V})$ is closed in Y [8]. Note that $V \cap \pi_X(\tilde{V}) = \emptyset$ since $\pi_X^{-1}(V) \cap \tilde{V} = \emptyset$. Let $\mathfrak{I} \subset \mathcal{O}(Y)$ be the ideal which defines $\pi_X(\tilde{V})$. Then $(f) + \mathfrak{I} \ni 1$ since $V \cap \pi_X(\tilde{V}) = \emptyset$. Hence there exists an $h \in \mathfrak{I}$ such that $h - 1 \in (f)$. Since $Y_h \subset Y - \pi_X(\tilde{V})$, $X_h = \pi_X^{-1}(Y_h) \subset \pi_X^{-1}(Y - \pi_X(\tilde{V})) \subset \tilde{U}$. Thus E is trivial over X_h. $\square$

We define an affine scheme $\tilde{Y} = \text{Spec}\,\tilde{A}$ by

$$\tilde{A} = \{h_1/h_2 | h_1, h_2 \in \mathcal{O}(Y), h_2 - 1 \in (f)\}.$$

Set $\tilde{Y}_f := Y_f \times_Y \tilde{Y}$, $\tilde{X} := \tilde{Y} \times_Y X$ and $\tilde{X}_f := \tilde{Y}_f \times_Y X$. The group of morphisms from X to $M := \mathrm{GL}(Q)$ is denoted by Mor (X, M) or $M(X)$. The group G acts on M by conjugation via the representation $\rho : G \to \mathrm{GL}(Q)$. The action of G on $M(X)$ is defined by $(g \cdot \mu)(x) = \rho(g)\mu(g^{-1}x)\rho(g)^{-1}$ for $g \in G$, $x \in X$, $\mu \in M(X)$. We denote the group of G-invariants of $M(X)$ by $\mathrm{Mor}\,(X, M)^G$ or $M(X)^G$. Let $[E] \in \mathrm{VEC}_G(X, Q)_0$. Then by the definition of $\mathrm{VEC}_G(X, Q)_0$, E has a trivialization over $\pi_X^{-1}(U) = X_f$, and by Lemma 1.1 E has a trivialization also over an open neighborhood of $\pi_X^{-1}(V)$, i.e., X_h for some $h \in \mathcal{O}(Y)$ such that $h - 1 \in (f)$. Hence, assigning to $[E]$ the transition function with respect to the trivializations $E|_{X_f} \cong X_f \times Q$ and $E|_{X_h} \cong X_h \times Q$, we have a bijection to a double coset (cf. [15, 3.4])

$$\mathrm{VEC}_G(X, Q)_0 \cong M(X_f)^G \backslash M(\tilde{X}_f)^G / M(\tilde{X})^G.$$

Since X^H is irreducible, the inclusion $X^H \hookrightarrow X$ induces an isomorphism $X^H /\!/ N(H) \xrightarrow{\sim} X /\!/ G$ where $N(H)$ is the normalizer of H in G [14]. Set $W := N(H)/H$. When we consider X^H as a W-variety, we denote it by B. Note that the principal isotropy group of B is trivial. Let $\beta : M(X)^G \to L(B)^W$ be the restriction map where $L := \mathrm{GL}(Q)^H$. We say X *has generically closed orbits* if $\pi_X^{-1}(\xi)$ for any $\xi \in Y_f$ consists of a closed orbit, i.e. $\pi_X^{-1}(\xi) \cong G/H$. When X has generically closed orbits, $GX_f^H = X_f$. Hence $M(X_f)^G = \mathrm{Mor}\,(GX_f^H, \mathrm{GL}(Q))^G \cong L(B_f)^W$, i.e., β is an isomorphsim over Y_f. The group homomorphism β induces a map

$$\begin{aligned} \mathrm{VEC}_G(X, Q)_0 \;&\cong\; M(X_f)^G \backslash M(\tilde{X}_f)^G / M(\tilde{X})^G \\ &\to\; L(B_f)^W \backslash L(\tilde{B}_f)^W / \beta(M(\tilde{X})^G), \end{aligned} \tag{1}$$

which is an isomorphism when X has generically closed orbits.

We decompose Q as an H-module

$$Q \cong \oplus_{i=1}^q n_i Q_i$$

where Q_i are pairwise non-isomorphic irreducible H-modules and n_i is the multiplicity of Q_i. We call Q *multiplicity free* with respect to H if $n_i = 1$ for all i. It follows from Schur's lemma that

$$L = \mathrm{GL}(Q)^H \cong \prod_{i=1}^q GL_{n_i}.$$

Let T be the center of L. Then T is W-stable and $T \cong (\mathbb{C}^*)^q$. When Q is multiplicity free with respect to H, $L = T$. Look at the action of W on T. Note that $g \in N(H)$ permutes the H-isotypic components n_iQ_i $(i = 1, \cdots, q)$. Since $w \in W$ acts on L by conjugation by $\rho(g)$ where $g \in N(H)$ is a representative of w, W acts on $T \cong (\mathbb{C}^*)^q$ by permuting $\mathbb{C}^*$s. Hence W acts on T as a subgroup of the symmetric group S_q via a continuous homomorphism from W to S_q. Thus the connected component W_0 of W containing the identity acts trivially on T and the action of W on T reduces to the action of W/W_0. The determinant map on each factor GL_{n_i} of L induces a homomorphism of groups; $\tau : L(B)^W \to T(B)^W$. The homomorphism τ induces a map

$$L(B_f)^W \backslash L(\tilde{B}_f)^W / \beta(M(\tilde{X})^G) \to T(\tilde{B}_f)^W / (T(B_f)^W (\tau \circ \beta) M(\tilde{X})^G). \quad (2)$$

By (1) and (2), we have

Lemma 1.2. *There exists a map*

$$\psi_{X,Q} : \mathrm{VEC}_G(X, Q)_0 \to T(\tilde{B}_f)^W / (T(B_f)^W (\tau \circ \beta) M(\tilde{X})^G).$$

If Q is multiplicity free with respect to H and X has generically closed orbits, then $\psi_{X,Q}$ is an isomorphism.

Remarks 1. For $t \in \mathcal{O}(Y)$, let $\mathrm{VEC}_G(X, Q; t)$ be the subset of $\mathrm{VEC}_G(X, Q)$ consisting of elements $[E]$ such that E is trivial over $\pi_X^{-1}(Y_t)$ and its complement. Then one obtains, in a similar way to the above, a map from $\mathrm{VEC}_G(X, Q; t)$ to a quotient group.

2. When H is trivial, $M = L$ and the target residue group in Lemma 1.2 is $\mathcal{O}(\tilde{Y}_f)^* / \mathcal{O}(Y_f)^* \tau(M(\tilde{X})^G)$, where $\mathcal{O}(\tilde{Y}_f)^*$ (resp. $\mathcal{O}(Y_f)^*$) denotes the group of invertible elements in $\mathcal{O}(\tilde{Y}_f)$ (resp. $\mathcal{O}(Y_f)$). If Q contains a trivial G-module, then $\tau = \det : M(\tilde{X})^G \to \mathcal{O}(\tilde{Y})^*$ is surjective. Furthermore, if $\mathrm{Pic}\, Y = (0)$, then the residue group $\mathcal{O}(\tilde{Y}_f)^* / \mathcal{O}(Y_f)^* \tau(M(\tilde{X})^G)$ becomes trivial (cf. proof of Lemma 1.3). Thus when H is trivial and $\mathrm{Pic}\, Y = (0)$ (e.g. X is a G-module with a trivial principal isotropy group), $\psi_{X,Q}$ becomes trivial if Q contains a trivial G-module.

We will analyse the target residue group in Lemma 1.2. We pose the following conditions:

(I) V is connected and $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$.
(II) The restriction $\mathcal{O}(\pi_B^{-1}(V))^* \to \mathcal{O}(X^G)^*$ is an isomorphism.

It follows from the conditions (I) and (II) that the restriction of $\pi_B^{-1}(V)$ onto X^G induces an isomorphism $T(\pi_B^{-1}(V))^W \cong T^W(X^G) \cong T^W$. Set

$$\begin{aligned} T(\tilde{B})_1 &:= \{\mu \in T(\tilde{B}) \mid \mu|_{\pi_B^{-1}(V)} = I\} \\ T(\tilde{B})_1^W &:= T(\tilde{B})_1 \cap T(\tilde{B})^W \end{aligned}$$

where I is the constant map to the identity element of T. Note that $T(\tilde{B}) = T(\tilde{B})_1 T(\pi_B^{-1}(V)) = T(\tilde{B})_1 T$.

Lemma 1.3. *Suppose that the conditions (I) and (II) are satisfied. If* $\operatorname{Pic} B = (0)$ *and* $\mathcal{O}(B)^* = \mathbb{C}^*$, *then*

$$T(\tilde{B}_f)^W = T(B_f)^W T(\tilde{B})_1^W.$$

Proof. We first claim that $T(\tilde{B}_f) = T(B_f)T(\tilde{B})_1$. Since $T(\tilde{B}) = T(\tilde{B})_1 T$, it suffices to prove $T(\tilde{B}_f) = T(B_f)T(\tilde{B})$. Note that every element of $T(\tilde{B}_f)$ is considered as a transition function of a Whitney sum of line bundles over B with respect to trivializations over B_f and an open neighborhood of $\pi_B^{-1}(V)$. Since $\operatorname{Pic} B = (0)$, every line bundle over B is trivial. This implies that $T(\tilde{B}_f) = T(B_f)T(\tilde{B})$. Let $\mu \in T(\tilde{B}_f)^W$. Write $\mu = \dot{\mu}\tilde{\mu}$ with $\dot{\mu} \in T(B_f)$ and $\tilde{\mu} \in T(\tilde{B})_1$. Note that $T(B_f) \cap T(\tilde{B})_1 = T(B)_1 = \{I\}$ since $\mathcal{O}(B)^* = \mathbb{C}^*$. Since μ is W-invariant, we have $\dot{\mu}^{-1}(w \cdot \dot{\mu}) = \tilde{\mu}(w \cdot \tilde{\mu})^{-1} \in T(B_f) \cap T(\tilde{B})_1 = \{I\}$ for every $w \in W$. Hence $\dot{\mu}$ and $\tilde{\mu}$ are W-invariant, and the assertion is thus verified. □

Set

$$M(\tilde{X})_1^G := \{\mu \in M(\tilde{X})^G \mid \mu|_{X^G} = I\}.$$

Note that $(\tau \circ \beta)(M(\tilde{X})_1^G) \subset T(\tilde{B})_1^W$ under the conditions (I) and (II).

Lemma 1.4. *Suppose that the assumptions in Lemma 1.3 are satisfied. If there exists a G-equivariant morphism $r : X \to X^G$ such that $r \circ i = id$ where $i : X^G \hookrightarrow X$ is the inclusion, then there exists an isomorphism*

$$T(\tilde{B}_f)^W / (T(B_f)^W (\tau \circ \beta) M(\tilde{X})^G) \cong T(\tilde{B})_1^W / (\tau \circ \beta)(M(\tilde{X})_1^G).$$

Proof. We claim that $(\tau \circ \beta) M(\tilde{X})^G \subset (\tau \circ \beta)(M(\tilde{X})_1^G) T^W$. In fact, let $\mu \in M(\tilde{X})^G$ and $\mu_0 := \mu|_{X^G} \in M^G(X^G)$. Then $(\tau \circ \beta)\mu_0 \in T^W(X^G) \cong T^W$. Let $p : M^G(X^G) \to M(X)^G$ be the group homomorphism induced by r. Then $\tilde{\mu} := p(\mu_0) \in M(X)^G$ satisfies $\tilde{\mu}|_{X^G} = \mu_0$. Since $\mathcal{O}(B)^* = \mathbb{C}^*$, $(\tau \circ \beta)\tilde{\mu} \in T^W$. The claim follows from that $\mu = \mu_1 \tilde{\mu}$ where $\mu_1 = \mu \tilde{\mu}^{-1} \in M(\tilde{X})_1^G$. Since $(\tau \circ \beta) M(\tilde{X})^G \subset (\tau \circ \beta)(M(\tilde{X})_1^G) T^W$ and $T(B_f)^W \cap T(\tilde{B})_1^W = T(B)_1^W = \{I\}$, we obtain by Lemma 1.3 the desired isomorphism. □

We proceed to analyse the residue group $T(\tilde{B})_1^W / (\tau \circ \beta)(M(\tilde{X})_1^G)$.

Let $\hat{Y}$ be the completion of Y along V and let $\hat{B} = \hat{Y} \times_Y B$ and $\hat{X} = \hat{Y} \times_Y X$. Note that an element of $M(\hat{X})^G$ (resp. $T(\hat{B})^W$) is considered as an invertible matrix (resp. an invertible diagonal matrix) with entries in $\mathcal{O}(\hat{X})$ (resp. $\mathcal{O}(\hat{B})$) invariant under the G-action (resp. the W-action). For $r \geq 1$, we define

$$\begin{aligned} T(\hat{B})^W_r &:= \{\mu \in T(\hat{B})^W \mid \mu = I \mod \mathfrak{b}^r\mathcal{O}(\hat{B})\} \\ M(\hat{X})^G_r &:= \{\mu \in M(\hat{X})^G \mid \mu = I \mod \mathfrak{a}^r\mathcal{O}(\hat{X})\}, \end{aligned}$$

where $\mathfrak{a} \subset \mathcal{O}(X)$ denotes the ideal of $X^G \subset X$ and $\mathfrak{b} \subset \mathcal{O}(B)$ denotes the ideal of $\pi_B^{-1}(V)$, i.e. $\mathfrak{b} = \sqrt{(f)}$. We define $L(\hat{B})^W_r$, similarly. Then there exists a canonical map

$$T(\tilde{B})^W_1/(\tau \circ \beta)(M(\tilde{X})^G_1) \to T(\hat{B})^W_1/(\tau \circ \beta)(M(\hat{X})^G_1).$$

We will show that this canonical map is a surjection when X has generically closed orbits. First, we prove

Lemma 1.5. *For every $r \geq 1$,*

$$T(\hat{B})^W_1 = T(\tilde{B})^W_1 T(\hat{B})^W_r.$$

Proof. It is clear that $T(\hat{B})^W_1 \supset T(\tilde{B})^W_1 T(\hat{B})^W_r$. We show the opposite inclusion. Let $\mu = (\mu_1(x), \ldots, \mu_q(x)) \in T(\hat{B})^W_1$ where $\mu_i(x) \in \mathcal{O}(\hat{B})$ and $\mu_i = 1 \mod \mathfrak{b}\mathcal{O}(\hat{B})$. Recall that W acts on $T \cong (\mathbb{C}^*)^q$ by permuting $\mathbb{C}^*$s. Since the identity component W_0 acts trivially on T, $\mu_i(x) \in \mathcal{O}(\hat{B})^{W_0}$ for $1 \leq i \leq q$. Let $\bar{\mu}_i(x) \in \mathcal{O}(B)^{W_0}$ be a function such that $\mu_i(x) = \bar{\mu}_i(x) \mod \mathfrak{b}^r\mathcal{O}(\hat{B})$. Since $\mu_i = 1 \mod \mathfrak{b}\mathcal{O}(\hat{B})$, $\bar{\mu}_i = 1 \mod \mathfrak{b}$. Define $\bar{\mu} := (\bar{\mu}_1(x), \ldots, \bar{\mu}_q(x))$ and $\tilde{\mu} := \prod_{w \in W/W_0} w \cdot \bar{\mu}$. Then $\tilde{\mu} \in T(\tilde{B})^W_1$ and $\tilde{\mu}^{-1}\mu \in T(\hat{B})^W_r$. □

Let $\mathfrak{m}$, $\mathfrak{l}$ and $\mathfrak{t}$ be the Lie algebras of M, L and T, respectively. Then $\mathfrak{m} = \text{End}\, Q$, $\mathfrak{l} = \text{End}\,(Q)^H \cong \oplus_{i=1}^q M_{n_i}$ and $\mathfrak{t} \cong \mathbb{C}^q$ where M_{n_i} denotes an $(n_i \times n_i)$-matrix. Let $\beta_* : \mathfrak{m}(X)^G \to \mathfrak{l}(B)^W$ be the homomorphism of $\mathcal{O}(Y)$-modules induced by the restriction of X onto B. Similarly, let $\tau_* : \mathfrak{l}(B)^W \to \mathfrak{t}(B)^W$ be the homomorphism induced by the trace map on each M_{n_i} of $\mathfrak{l} \cong \oplus_{i=1}^q M_{n_i}$. Note that

$$\mathfrak{t}(B)^W \cong (\mathcal{O}(B) \otimes_{\mathbb{C}} \mathfrak{t})^W, \qquad \mathfrak{l}(B)^W \cong (\mathcal{O}(B) \otimes_{\mathbb{C}} \mathfrak{l})^W$$
$$\text{and} \quad \mathfrak{m}(X)^G \cong (\mathcal{O}(X) \otimes_{\mathbb{C}} \mathfrak{m})^G,$$

which are all finitely generated modules over $\mathcal{O}(Y)$ (cf. [8, II,3.2]). For a positive integer r, we define

$$\begin{aligned} \mathfrak{t}(B)_r^W &:= (\mathfrak{b}^r \otimes_{\mathbb{C}} \mathfrak{t})^W, \\ \mathfrak{l}(B)_r^W &:= (\mathfrak{b}^r \otimes_{\mathbb{C}} \mathfrak{l})^W, \\ \mathfrak{m}(X)_r^G &:= (\mathfrak{a}^r \otimes_{\mathbb{C}} \mathfrak{m})^G, \end{aligned}$$

which are also finitely generated modules over $\mathcal{O}(Y)$. We define $\mathfrak{t}(\hat{B})_r^W$, $\mathfrak{l}(\hat{B})_r^W$ and $\mathfrak{m}(\hat{X})_r^G$, similarly. The exponentials $\exp : \mathfrak{l} \to L$ and $\exp : \mathfrak{m} \to M$ induce isomorphisms (with inverse log) $\mathfrak{l}(\hat{B})_r^W \xrightarrow{\sim} L(\hat{B})_r^W$ and $\mathfrak{m}(\hat{X})_r^G \xrightarrow{\sim} M(\hat{X})_r^G$ (Here, the latter exponential series converges in the $\mathfrak{a}$-adic topology).

Lemma 1.6. *Suppose that X has generically closed orbits. Then there exists an integer r_0 such that $\beta_*\mathfrak{m}(X)_1^G \supset \mathfrak{l}(B)_r^W$ and $\beta(M(\hat{X})_1^G) \supset L(\hat{B})_r^W$ for all $r \geq r_0$.*

Proof. Let $\{C_i\}$ and $\{A_j\}$ be generating systems of $\mathfrak{l}(B)_1^W$ and $\mathfrak{m}(X)_1^G$ over $\mathcal{O}(Y)$, respectively. Since X has generically closed orbits, $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is an isomorphism. Thus C_i is written as $C_i = \beta_*(\sum_j c_{ij}A_j)$ where $c_{ij} \in \mathcal{O}(Y)_f$. Let $e_{ij} \geq 0$ be the minimal integer such that $f^{e_{ij}}c_{ij} \in \mathcal{O}(Y)$ and d be the minimal integer such that $\mathfrak{b}^d \subseteq (f)$. Put $e := \max_{i,j}\{e_{ij}\}$ and $r_0 := de + 1$. Then for $r \geq r_0$, any element of $\mathfrak{l}(B)_r^W$ is of the form f^eC where $C \in \mathfrak{l}(B)_1^W$. Since $C = \sum_i c_iC_i$ for $c_i \in \mathcal{O}(Y)$ and $f^e c_{ij} \in \mathcal{O}(Y)$ for every i, j, so $f^eC \in \beta_*\mathfrak{m}(X)_1^G$. Hence $\beta_*\mathfrak{m}(X)_1^G \supset \mathfrak{l}(B)_r^W$. The second inclusion follows from $\beta_*\mathfrak{m}(\hat{X})_1^G \supset \mathfrak{l}(\hat{B})_r^W$ via the exponential maps. □

Remark In order to prove Lemma 1.6, it is sufficient to hold that $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is surjective.

Since $\tau_* : \mathfrak{l}(\hat{B})_r^W \to \mathfrak{t}(\hat{B})_r^W$ is the trace map, τ_* is surjective. Hence, via the exponential maps, $T(\hat{B})_r^W = \tau(L(\hat{B})_r^W)$. Under the assumption in Lemma 1.6, $T(\hat{B})_r^W = \tau(L(\hat{B})_r^W) \subset (\tau \circ \beta)(M(\hat{X})_1^G)$ for a sufficiently large r. By this together with Lemma 1.5, we obtain

Lemma 1.7. *Suppose that X has generically closed orbits. Then the canonical map*

$$T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{X})_1^G) \to T(\hat{B})_1^W/(\tau \circ \beta)(M(\hat{X})_1^G)$$

is a surjection. Furthermore, if Q is multiplicity free with respect to H, then $L(\tilde{B})_1^W/\beta(M(\tilde{X})_1^G) \to L(\hat{B})_1^W/\beta(M(\hat{X})_1^G)$ is an isomorphism.

Proof. The first assertion is clear from the above statement. As for the second assertion, it suffices to show that the canonical map is injective. We will show that $\beta(M(\hat{X})_1^G) \cap L(\tilde{B})_1^W \subset \beta(M(\tilde{X})_1^G)$. Let $\hat{D} \in M(\hat{X})_1^G$ and $\beta(\hat{D}) \in \beta(M(\hat{X})_1^G) \cap L(\tilde{B})_1^W$. We regard $\hat{D}$ as an element of $\mathfrak{m}(\hat{X})^G$ and show that $\hat{D} \in \mathfrak{m}(\tilde{X})^G$. Since $\beta(\hat{D}) \in L(\tilde{B})_1^W$ is translated as $\beta_*(\hat{D}) \in \mathfrak{l}(\tilde{B})^W$, it follows from Lemma 1.6 that $f^r\beta_*(\hat{D}) = \beta_*(\tilde{D})$ for a sufficiently large r and $\tilde{D} \in \mathfrak{m}(\tilde{X})^G$. Since X has generically closed orbits, $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is an isomorphism, so, $\beta_* : \mathfrak{m}(X)^G \to \mathfrak{l}(B)^W$ is an injection. Hence $\beta_* : \mathfrak{m}(\hat{X})^G \to \mathfrak{l}(\hat{B})^W$ is also an injection. Thus $f^r\hat{D} = \tilde{D}$. This implies that $\hat{D} \in \mathfrak{m}(\tilde{X})^G$. Hence $\hat{D} \in M(\tilde{X})_1^G$ and the assertion follows. □

The logarithmic map induces an isomorphism

$$T(\hat{B})_1^W/(\tau \circ \beta)(M(\hat{X})_1^G) \cong \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G.$$

We set

$$C_X(Q) := \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G.$$

When Q is multiplicity free with respect to H, $C_X(Q) = \mathfrak{l}(\hat{B})_1^W/\beta_*\mathfrak{m}(\hat{X})_1^G$.

By the results obtained so far, we have

Theorem 1.8. *There exists a map*

$$T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{X})_1^G) \to \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G = C_X(Q),$$

which is an isomorphism when Q is multiplicity free with respect to H and X has generically closed orbits.

2. G-vector bundles over G-modules

In this section, we consider the case where the base space X is a G-module P and give a proof of Theorem 1 in the introduction. Let P be a G-module such that $Y = P//G$ is of dimension ≥ 1 and the ideal of $V = Y - U$ is principal. Note that the ideal of V is generated by an invariant homogeneous polynomial $f \in \mathcal{O}(P)^G$ and that V is connected. Let H be a principal isotropy group of P and let $B = P^H$.

Lemma 2.1. (1) Pic $B = (0)$ *and* $\mathcal{O}(B)^* = \mathcal{O}(P^G)^* = \mathbb{C}^*$.
(2) $\pi_B^{-1}(V)$ *is a connected affine cone and* $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$.

Proof. (1) The assertion follows from the fact that B and P^G are affine spaces.

(2) One easily sees that $\pi_B^{-1}(V)$ is a connected affine cone. Indeed, $\pi_B^{-1}(V)$ is a union of irreducible reduced affine cones $\cup_j \mathrm{Spec}\, R^{(j)}$ passing through the origin. Each affine cone Spec $R^{(j)}$ has a positively graded integral domain $R^{(j)} = \oplus_{k\geq 0} R_k^{(j)}$ as the coordinate ring such

that $R_0^{(j)} = \mathbb{C}$. Since $(R^{(j)})^* = \mathbb{C}^*$ for each j, the standard argument in commutative algebras shows that $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$. □

The projection $p : P \to P^G$ is G-equivariant and has the property $p \circ i = id$ for the inclusion $i : P^G \hookrightarrow P$. By this fact and the results obtained so far, we obtain a map $\Psi_{P,Q}$ for a G-module Q;

$$\begin{array}{lcll} \mathrm{VEC}_G(P,Q)_0 & \stackrel{\psi_{P,Q}}{\to} & T(\tilde{B}_f)^W/(T(B_f)^W(\tau\circ\beta)M(\tilde{P})^G) & \text{(Lemma 1.2)} \\ & \cong & T(\tilde{B})_1^W/(\tau\circ\beta)(M(\tilde{P})_1^G) & \text{(Lemmas 1.4, 2.1)} \\ & \to & \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{P})_1^G = C_P(Q) & \text{(Theorem 1.8).} \end{array}$$

Hence we have

Theorem 2.2. *Let P be a G-module as above and let Q be a G-module. There is a map*

$$\Psi_{P,Q} : \mathrm{VEC}_G(P,Q)_0 \to C_P(Q)$$

which is an isomorphism when Q is multiplicity free with respect to H and P has generically closed orbits.

Remarks 1. Let P be any G-module and let t be a G-invariant homogeneous polynomial on P. We use the notation in the remark of Lemma 1.2. By the construction similar to the above, one obtains a map

$$\Psi_{P,Q}(t) : \mathrm{VEC}_G(P,Q;t) \to \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{P})_1^G =: C_{P,t}(Q)$$

where the completion is (t)-adic completion. One can show that $\Psi_{P,Q}(t)$ is surjective for any G-module Q if one takes $t \in \mathcal{O}(Y)$ so that Y_t is contained in the principal stratum of $\overline{GP^H}$ (cf. [15, 1.1], [2, 6.5]).

2. When H is trivial and Q contains a trivial G-module, $\psi_{P,Q}$ is trivial (remark of Lemma 1.2), hence $\Psi_{P,Q}$ is also trivial.

This completes the proof of Theorem 1 in the introduction except the statement on $C_P(Q)$. Note that Theorem 1 holds also in the case $\dim P/\!/G = 1$. When $\dim P/\!/G = 1$, it is known that $P/\!/G \cong \mathbb{A}^1$ and $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q) = \mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q)_0$ for $m \geq 0$ ([11], [16]). Suppose that $\dim P/\!/G = 1$. Then $C_P(Q)$ is a finite $\mathbb{C}$-module by the formula (3) below (cf. Lemma 2.3) and $C_{P\oplus\mathbb{C}^m}(Q) \cong (\mathbb{C}[y_1, \cdots, y_m])^p$ by easy calculation. By comparing $\Psi_{P\oplus\mathbb{C}^m,Q}$ with the isomorphism $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q) \xrightarrow{\sim} (\mathbb{C}[y_1, \cdots, y_m])^p$ given in [16] (cf. [11]), one sees that $\Psi_{P\oplus\mathbb{C}^m,Q}$ for $m \geq 0$ is an isomorphism for any P and Q.

Now, we look at $C_P(Q)$ more closely. A G-module P is called *cofree* if $\mathcal{O}(P)$ is a free module over $\mathcal{O}(P)^G$. It is known that cofree modules are coregular, i.e., $P/\!/G$ is isomorphic to affine space (cf. [24]). Furthermore, if P^H is a cofree $N(H)$-module, then P is a cofree G-module

[24]. We suppose that B is a cofree W-module and make some obeservation on $C_P(Q)$. Then, $\mathcal{O}(Y)$ is isomorphic to a polynomial ring and $\mathfrak{m}(P)^G$ and $\mathfrak{t}(B)^W$ are finite free modules over $\mathcal{O}(Y)$. Since $\mathfrak{b}$ is principal, $\mathfrak{t}(B)^W_1$ is also a finite free module over $\mathcal{O}(Y)$. The rank of $\mathfrak{t}(B)^W_1$ is the same as the rank of $\mathfrak{t}(B)^W$, which is equal to $q = \dim \mathfrak{t}$ [24]. Note that $\mathcal{O}(Y)$, $\mathfrak{m}(P)^G$ and $\mathfrak{t}(B)^W$ inherit a grading on $\mathcal{O}(P)$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals, $\mathfrak{m}(P)^G_1$ and $\mathfrak{t}(B)^W_1$ are also graded. Let $\{A_i; 1 \le i \le \ell\}$ be a homogeneous generating system of $\mathfrak{m}(P)^G_1$ over $\mathcal{O}(Y)$ and let $\{C_i; 1 \le i \le q\}$ be a homogeneous basis of $\mathfrak{t}(B)^W_1$ over $\mathcal{O}(Y)$. Then

$$\tau_*\beta_* A_i = \sum_{j=1}^{q} a_{ij} C_j \quad \text{for} \quad a_{ij} \in \mathcal{O}(Y).$$

Noting that $\mathfrak{t}(\hat{B})^W_1 = \mathfrak{t}(B)^W_1 \otimes_{\mathcal{O}(Y)} \mathcal{O}(\hat{Y})$ and $\mathfrak{m}(\hat{P})^G_1 = \mathfrak{m}(P)^G_1 \otimes_{\mathcal{O}(Y)} \mathcal{O}(\hat{Y})$,

$$C_P(Q) \cong \oplus_{j=1}^{q} \mathcal{O}(\hat{Y})/\widehat{\mathfrak{a}_j} \tag{3}$$

where $\widehat{\mathfrak{a}_j} = \mathfrak{a}_j \mathcal{O}(\hat{Y})$ and $\mathfrak{a}_j$ is the ideal in $\mathcal{O}(Y)$ generated by $\{a_{ij}; 1 \le i \le \ell\}$. Let $e_j = \deg C_j$ and $a_i = \deg A_i$. Since τ_* and β_* preserve the grading, $\deg a_{ij} = a_i - e_j$ if $a_{ij} \neq 0$. The following is easily proved.

Lemma 2.3. *Suppose that B is cofree. If there is some j such that $a_i > e_j$ for any i, then $C_P(Q)$ is non-trivial. If there exists some j such that* ht $\mathfrak{a}_j < \dim Y$*, then $C_P(Q)$ is an infinite dimensional $\mathbb{C}$-module.*

Remark The module $C_P(Q)$ can be of infinite dimension, but of countably-infinite dimension.

This completes the proof of Theorem 1. By Theorem 2.2 and Lemma 2.3, we have

Corollary 2.4. *Suppose that $\Psi_{P,Q}$ in Theorem 2.2 is surjective and B is cofree. If $a_i > e_j$ for some j and any i, then* $\mathrm{VEC}_G(P,Q)_0$ *is non-trivial. If there exists some j such that* ht $\mathfrak{a}_j < \dim Y$*, then* $\mathrm{VEC}_G(P,Q)_0$ *contains an infinite dimensional space.*

We give a couple of examples.

Example 2.1

Let $G = SL_n$ $(n \ge 2)$ and let P be the Lie algebra $\mathfrak{sl}_n$ with adjoint action. We denote a maximal torus of G by T_n and its Lie algebra by $\mathfrak{t}_n$. Then the principal isotropy group of $\mathfrak{sl}_n$ is T_n and $B = (\mathfrak{sl}_n)^{T_n} = \mathfrak{t}_n$. $W = N(T_n)/T_n$ is the Weyl group which is isomorphic to S_n. The algebraic quotient space Y is $\mathfrak{sl}_n//G \cong \mathfrak{t}_n//W \cong \mathbb{A}^{n-1}$ and V is of codimension one. Hence the ideal of V is generated by a single homogeneous polynomial $f \in \mathcal{O}(Y) \cong \mathbb{C}[t_1, \cdots, t_{n-1}]$. Since the general fiber of the quotient map of $\mathfrak{sl}_n$ is isomorphic to G/T_n, $\mathfrak{sl}_n$ has generically closed orbits. Let φ_1 be the standard representation space of G and φ_1^m ($m \geq 1$) be the symmetric tensor product $S^m(\varphi_1)$. Let $Q = \varphi_1^m$. Then Q is multiplicity free with respect to T_n. Hence $L = T \cong (\mathbb{C}^*)^q$ for $q = \dim Q = \binom{n+m-1}{m}$.

Consider the case $n = 2$. Then $G = SL_2$ and the quotient map is given by the determinant map $t : P = \mathfrak{sl}_2 \to \mathfrak{sl}_2//G \cong \mathbb{A}^1$. Hence $\mathcal{O}(Y) = \mathbb{C}[t]$ and t is, as an element of $\mathcal{O}(B)^W$, written as $t = x^2$ with a coordinate x on $B = \mathfrak{t}_2 \cong \mathbb{C}$. Note that $T_2 \cong \mathbb{C}^*$ and $W \cong \mathbb{Z}/2\mathbb{Z}$. The stratification of $\mathfrak{sl}_2//G = \mathbb{A}^1$ consists of two strata, $\{0\}$ and $\mathbb{A}^1 - \{0\}$. Hence $V = \{0\}$ and $f = t$. Let R_m be the SL_2-module of binary forms of degree m. Then $P = \mathfrak{sl}_2 \cong R_2$ and $Q \cong R_m$. As a $T_2 = \mathbb{C}^*$-module, $Q = \oplus_{l=0}^m Q_{m-2l}$ where Q_{m-2l} is an irreducible T_2-module with weight $m-2l$. As a G-module, $\mathfrak{m} = \operatorname{End} R_m \cong (R_m)^* \otimes R_m \cong \oplus_{l=0}^m R_{2l}$. Hence,

$$\mathfrak{m}(\mathfrak{sl}_2)^G \cong \oplus_{l=0}^m (\mathcal{O}(R_2) \otimes R_{2l})^G = \oplus_{l=0}^m M_l$$

and

$$\mathfrak{l}(\mathfrak{t}_2)^W \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{t}_2) \otimes R_{2l}^{T_2})^W = \oplus_{l=0}^m N_l$$

where $M_l := (\mathcal{O}(R_2) \otimes R_{2l})^G$ and $N_l := (\mathcal{O}(\mathfrak{t}_2) \otimes R_{2l}^{T_2})^W$. The modules M_l and N_l are free over $\mathcal{O}(Y) = \mathbb{C}[t]$ of rank one. In fact, since $M_l \cong \operatorname{Mor}(R_2, R_{2l})^G$, the homogeneous generator A_l of M_l is given by the l-th power map and the homogeneous generator C_l of $N_l = (\mathbb{C}[x] \otimes R_{2l}^{T_2})^W$ is given by $1 \otimes e_l$ for l even, $x \otimes e_l$ for l odd, where e_l is a base of $R_{2l}^{T_2} \cong \mathbb{C}$. Hence $\mathfrak{m}(\mathfrak{sl}_2)^G$ and $\mathfrak{l}(\mathfrak{t}_2)^W$ are free modules over $\mathbb{C}[t]$ of rank $m+1$. Note that $\deg A_l = l$ and $\deg C_l$ is 0 for l even, 1 for l odd. Since $\mathbb{C}[t]$ is a principal ideal domain, $\mathfrak{m}(\mathfrak{sl}_2)_1^G$ is also free over $\mathbb{C}[t]$. A homogeneous basis of $\mathfrak{m}(\mathfrak{sl}_2)_1^G$ over $\mathbb{C}[t]$ is $\{tA_0, A_l; l = 1, 2, \cdots, m\}$ since $\mathfrak{sl}_2{}^G = \{O\}$. Since $\mathfrak{b} = \sqrt{(t)} = (x)$, a homogeneous basis of $\mathfrak{l}(\mathfrak{t}_2)_1^W$ over $\mathbb{C}[t]$ is $\{tC_0, tC_{2l}, C_{2l-1}; l = 1, \cdots, m/2\}$ for m even, $\{tC_{2l}, C_{2l+1}; l = 0, 1, \cdots, [m/2]\}$ for m odd. Here, $[a]$ denotes the largest integer not-exceeding a. Since $\beta_*(A_l) = t^{[l/2]}C_l$,

$$C_{\mathfrak{sl}_2}(\varphi_1^m) \cong \mathfrak{l}(\mathfrak{t}_2)_1^W / \beta_* \mathfrak{m}(\mathfrak{sl}_2)_1^G \cong \mathbb{C}^p$$

where $p = \sum_{l=1}^m [(l-1)/2] = [(m-1)^2/4]$. Since it follows from $\mathfrak{sl}_2//G \cong \mathbb{A}^1$ that $\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) = \mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m)_0$, we have by Theorem 2.2

Proposition 2.5. [23] *Let $G = SL_2$. Then*

$$\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) \cong \mathbb{C}^p \qquad \textit{for} \quad p = [(m-1)^2/4].$$

Next, consider the case that $n \geq 3$. As a G-module,

$$\mathfrak{m} = \operatorname{End} \varphi_1^m \cong (\varphi_1^m)^* \otimes \varphi_1^m \cong \oplus_{l=0}^m \mathfrak{sl}_n^l$$

where $\mathfrak{sl}_n^l$ is the irreducible component of the highest weight in $S^l(\mathfrak{sl}_n)$. Hence

$$\mathfrak{m}(\mathfrak{sl}_n)^G \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G = \oplus_{l=0}^m M_l$$

where $M_l := (\mathcal{O}(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G$. Similarly,

$$\mathfrak{l}(\mathfrak{t}_n)^W \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{t}_n) \otimes (\mathfrak{sl}_n^l)^{T_n})^W = \oplus_{l=0}^m N_l$$

where $N_l := (\mathcal{O}(\mathfrak{t}_n) \otimes (\mathfrak{sl}_n^l)^{T_n})^W$. It is known that $\mathfrak{t}_n$ is cofree (cf. [24]). Thus M_l and N_l, hence $\mathfrak{m}(\mathfrak{sl}_n)^G$ and $\mathfrak{l}(\mathfrak{t}_n)^W$, are finite free modules over $\mathcal{O}(Y)$. Since $\mathcal{O}(\mathfrak{sl}_n) \cong \oplus_{d\geq 0} S^d(\mathfrak{sl}_n)$, $M_l \cong \oplus_{d \geq 0}(S^d(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G$. Hence every homogeneous generator of M_l has degree $\geq l$. The homomorphism $\beta_* : \mathfrak{m}(\mathfrak{sl}_n)^G \to \mathfrak{l}(\mathfrak{t}_n)^W$ maps M_l to N_l. Set $M(1)_l := (\mathfrak{a} \otimes \mathfrak{sl}_n^l)^G$ and $N(1)_l := (\mathfrak{b} \otimes (\mathfrak{sl}_n^l)^{T_n})^W$. Then $\mathfrak{m}(\mathfrak{sl}_n)_1^G = \oplus_{l=0}^m M(1)_l$ and $\mathfrak{l}(\mathfrak{t}_n)_1^W = \oplus_{l=0}^m N(1)_l$. The homomorphism β_* maps $M(1)_l$ to $N(1)_l$. Let $\{A_i\}$ be a homogeneous generating system of $M(1)_m$ over $\mathcal{O}(Y)$ and $\{C_i\}$ be a homogenous basis of $N(1)_m$ over $\mathcal{O}(Y)$. Then $\beta_*(A_i) = \sum_j a_{ij} C_j$ for $a_{ij} \in \mathcal{O}(Y)$. Since $\deg A_i \geq m$ for all i and $\deg C_j < |W| + \deg f$ [8, II,3.6], $\deg a_{ij} > 0$ if m is sufficiently large. Hence $N(1)_m / \beta_*(M(1)_m)$ is non-trivial for $m \gg 0$. We have by Theorem 2.2;

Proposition 2.6. (cf. [6]) *Let $n \geq 3$ and $G = SL_n$. For $m \geq 1$, $\mathrm{VEC}_G(\mathfrak{sl}_n, \varphi_1^m)_0 \cong C_{\mathfrak{sl}_n}(\varphi_1^m)$. In particular, $\mathrm{VEC}_G(\mathfrak{sl}_n, \varphi_1^m)_0$ is non-trivial for a sufficiently large m.*

Remark In order to show that $C_{\mathfrak{sl}_n}(\varphi_1^m)$ contains an infinite dimensional module for $n \geq 3$, we need to prove that the height of the ideal $\mathfrak{a}_j$ generated by $a_{ij} \in \mathcal{O}(Y)$ (cf. Lemma 2.3) is smaller than $n-1$. However, to calculate generators of $N(1)_l$ and $M(1)_l$ by hand is a hard job.

Next is a new example of $\mathrm{VEC}_G(P,Q)_0$ containg an infinite dimensional space.

Example 2.2

Let $P = P_1 \oplus P_2$ and $G = G_1 \times G_2$ where P_i is a G_i-module with one-dimensional quotient for $i = 1, 2$. Then P is a G-module with trivial G_i-actions on P_j for $i \neq j$ and $P/\!/G \cong \mathbb{A}^2$. A principal isotropy group H of P is $H_1 \times H_2$ where H_i is a principal isotropy group of P_i. The complement of the principal stratum in $P/\!/G \cong \mathbb{A}^2$ is a union of two lines. Let Q_i $(i = 1, 2)$ be a G_i-module. By the statement below Theorem 2.2, there are isomorphisms $\mathrm{VEC}_{G_i}(P_i, Q_i) \cong C_{P_i}(Q_i) \cong \mathbb{C}^{p_i}$ for $i = 1, 2$. Let $Q = Q_1 \oplus Q_2$. Then Q is multiplicity free with respect to H when Q_i is multiplicity free with respect to H_i for $i = 1, 2$ and $\dim(Q_1^{H_1} \oplus Q_2^{H_2}) \le 1$. In this case, $C_P(Q)$ is easily computed and isomorphic to $\mathbb{C}[u_1]^{p_2} \oplus \mathbb{C}[u_2]^{p_1}$ where $\mathcal{O}(P_1)^{G_1} = \mathbb{C}[u_1]$ and $\mathcal{O}(P_2)^{G_2} = \mathbb{C}[u_2]$. By Theorem 2.2, we have with the above notation

Theorem 2.7. *Suppose that Q_i is multiplicity free with respect to H_i for $i = 1, 2$ and $\dim(Q_1^{H_1} \oplus Q_2^{H_2}) \le 1$. Then there is a map*

$$\mathrm{VEC}_G(P_1 \oplus P_2, Q_1 \oplus Q_2)_0 \to \mathbb{C}[u_1]^{p_2} \oplus \mathbb{C}[u_2]^{p_1},$$

which is an isomorphism when P_i has generically closed orbits for $i = 1, 2$.

Remark One can show that the map in Theorem 2.7 is surjective for any Q and any P_i by using the fact that $Z_f = GP_f^H$ for $Z := \overline{GP^H}$ when $\mathcal{O}(P)^G = \mathbb{C}[f]$ (cf. [15, 1.1], the remark of Theorem 2.2).

Apply Theorem 2.7 to the case where $G = SL_2 \times SL_2$, $P = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and $Q = \varphi_1^m \oplus \varphi_1^n$. Since $\mathfrak{sl}_2$ has generically closed orbits and φ_1^m is multiplicity free with respect to a principal isotropy group of $\mathfrak{sl}_2$ for $m \ge 1$, we have

Theorem 2.8. *Let $G = SL_2 \times SL_2$. Then*

$$\mathrm{VEC}_G(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \varphi_1^m \oplus \varphi_1^n)_0 \cong \mathbb{C}[u_1]^{p(n)} \oplus \mathbb{C}[u_2]^{p(m)}.$$

Here $p(n) = [(n-1)^2/4]$ and either m or n is odd.

3. G-vector bundles over $G \times (\mathbb{Z}/d\mathbb{Z})$-varieties

In this section, we consider in the case that V is not connected. Such a case occurs when X is a G-stable affine quadric with fixpoints and one-dimensional quotient. As is remarked in the introduction, when G is connected, such an affine quadric X is G-isomorphic to an affine quadric

$$X_P = \{(x, v) \in P \oplus \mathbb{C} \mid u(x) + v^2 = 1\}$$

where P is an orthogonal G-module with $P//G \cong \mathbb{A}^1$ and $u(x)$ is an invariant quadratic form on P such that $\mathcal{O}(P)^G = \mathbb{C}[u]$. Recall that X_P is viewed as a $G \times (\mathbb{Z}/2\mathbb{Z})$-variety. We generalize this situation. Let P be anew a G-module as in section 2, i.e., P is a G-module such that $\dim P//G \ge 1$ and the ideal of the complement of the principal stratum in $P//G$ is generated by a homogeneous polynomial $f \in \mathcal{O}(P)^G$. For $d \ge 2$, define a G-stable hypersurface $X_P(d)$ as follows;

$$X_P(d) := \{(x, v) \in P \oplus \mathbb{C} \mid f(x) + v^d = 1\}.$$

Then the fixpoint locus $X_P(d)^G$ consists of d connected components. The complement V of the principal stratum in $X_P(d)//G$ has d connected components and each connected component of $\pi^{-1}_{X_P(d)}(V)$ contains one connected component of $X_P(d)^G$. A principal isotropy group H of $X_P(d)$ is a principal isotropy group of P. As in the case of affine quadrics, $X_P(d)$ has a $\mathbb{Z}/d\mathbb{Z}$-action induced by a (non-trivial) linear action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{C}$. Hence $X_P(d)$ is viewed as a $G \times (\mathbb{Z}/d\mathbb{Z})$-variety. Then $X_P(d)/(\mathbb{Z}/d\mathbb{Z})$ is G-isomorphic to P. Let $\pi_{\mathbb{Z}_d} : X_P(d) \to X_P(d)/(\mathbb{Z}/d\mathbb{Z}) \cong P$ be the quotient by $\mathbb{Z}/d\mathbb{Z}$. Let $[E] \in \mathrm{VEC}_G(P, Q)$ for a G-module Q. Then $\pi^*_{\mathbb{Z}_d} E$ is a $G \times (\mathbb{Z}/d\mathbb{Z})$-vector bundle over $X_P(d)$. Viewing $\pi^*_{\mathbb{Z}_d} E$ as a G-vector bundle, we obtain a map

$$\pi^*_{\mathbb{Z}_d} : \mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_G(X_P(d), Q).$$

Since $E \cong \pi^*_{\mathbb{Z}_d} E/(\mathbb{Z}/d\mathbb{Z})$ [9], we have

Lemma 3.1. *The map $\pi^*_{\mathbb{Z}_d}$ is injective.*

Note that $\pi^*_{\mathbb{Z}_d}$ maps $\mathrm{VEC}_G(P, Q)_0$ to $\mathrm{VEC}_G(X_P(d), Q)_0$. By Lemma 3.1 and Theorem 2.2, we obtain

Theorem 3.2. *The map $\pi^*_{\mathbb{Z}_d}$ induces an injection*

$$\mathrm{VEC}_G(P, Q)_0 \to \mathrm{VEC}_G(X_P(d), Q)_0.$$

Hence, if $\Psi_{P,Q}$ in Theorem 2.2 is a surjection onto a non-trivial $C_P(Q)$, then $\mathrm{VEC}_G(X_P(d), Q)_0$ is non-trivial.

If we take as an f in the definition of $X_P(d)$ any G-invariant polynomial on P, then we obtain Theorem 2 in the introduction.

Remark Theorem 3.2 is generalized as follows. Let P_i ($i = 1, 2$) be a G_i-module such that $\dim P_1//G_1 \geq 1$ and $\dim P_2//G_2 = 1$. Let t be a homogeneous generator of $\mathcal{O}(P_2)^{G_2}$. For $f \in \mathcal{O}(P_1)^{G_1}$, define a $G_1 \times G_2$-stable hypersurface $X(f)$ as follows:

$$X(f) := \{(x_1, x_2) \in P_1 \oplus P_2 \mid f(x_1) + t(x_2) = 1\}.$$

Then the quotient map $\pi_{G_2} : X(f) \to X(f)//G_2 \cong P_1$ induces an injection for a G_1-module Q

$$\pi^*_{G_2} : \mathrm{VEC}_{G_1}(P_1, Q) \to \mathrm{VEC}_{G_1}(X(f), Q).$$

Recall that $\mathrm{VEC}_G(P, Q)_0 = \mathrm{VEC}_G(P, Q) \cong \mathbb{C}^p$ when P has one-dimensional quotient. Hence we have by Theorem 3.2

Corollary 3.3. *Suppose that X_P is a G-stable affine quadric defined as above. Then $\mathrm{VEC}_G(X_P, Q)_0$ contains a space isomorphic to $\mathbb{C}^p$ where p is a nonnegative integer such that $\mathrm{VEC}_G(P, Q) \cong \mathbb{C}^p$.*

We give a couple of examples.

Example 3.1

Let $G = SL_2$. We use the same notation as in Example 2.1. Let $P = \mathfrak{sl}_2$ and $Q = \varphi_1^m$ for $m \geq 1$. Then $\mathcal{O}(\mathfrak{sl}_2)^G = \mathbb{C}[t]$ with an invariant polynomial t of degree 2 and $\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) \cong \mathbb{C}^p$ for $p = [\frac{(m-1)^2}{4}]$. Let X be a G-stable affine quadric $\{(x, v) \in \mathfrak{sl}_2 \oplus \mathbb{C} \mid t + v^2 = 1\}$. Then by Corollary 3.3,

Proposition 3.4. *With the above notation,* $\mathrm{VEC}_G(X, \varphi_1^m)_0$ *contains* $\mathbb{C}^p$ *for* $p = [\frac{(m-1)^2}{4}]$.

Remark It is known that $\mathrm{VEC}_G(\mathfrak{sl}_2 \oplus \mathbb{C}, \varphi_1^m)_0 \cong \mathbb{C}[v]^p$ by [16].

Example 3.2

Let $G = G_1 \times G_2$, $P = P_1 \oplus P_2$, and $Q = Q_1 \oplus Q_2$ as in Example 2.2. Let $\mathcal{O}(P_1)^{G_1} = \mathbb{C}[u_1]$ and $\mathcal{O}(P_2)^{G_2} = \mathbb{C}[u_2]$ where u_i is a G_i-invariant homogeneous polynomial on P_i. Then $P//G \cong \mathbb{A}^2 = \mathrm{Spec}\,\mathbb{C}[u_1, u_2]$ and the complement of the principal stratum is defined by $u_1 u_2 = 0$. We define for $d \geq 2$

$$X_d := \{(x_1, x_2, v) \in P_1 \oplus P_2 \oplus \mathbb{C} \mid u_1(x_1)u_2(x_2) + v^d = 1\}.$$

Then by the remark of Theorem 2.7 and Theorem 3.2,

Proposition 3.5. *Under the notation and the assumptions in Theorem 2.7,* $\mathrm{VEC}_G(X_d, Q_1 \oplus Q_2)_0$ *contains an infinite dimensional space if* $p_1 + p_2 > 0$.

Example 3.3

Let $G = SL_3$ and $P = \mathfrak{sl}_3$ with adjoint action. Then $P//G \cong \mathbb{A}^2$ and the complement of the principal stratum in $P//G$ is defined by an invariant homogeneous polynomial f of degree 6. For $d \geq 2$, define

$$X_d = \{(x, v) \in \mathfrak{sl}_3 \oplus \mathbb{C} \mid f + v^d = 1\}.$$

It is known that $\mathrm{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3)_0$ contains a space isomorphic to $\Omega^1_{\mathbb{C}}$ which is the module of Kähler differentials of $\mathbb{C}$ over $\mathbb{Q}$ [17]. Hence we have by Theorem 3.2

Proposition 3.6. $\mathrm{VEC}_G(X_d, \mathfrak{sl}_3)_0$ *contains an uncountably-infinite dimensional space.*

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