SURJECTIVE DERIVATIONS IN SMALL DIMENSIONS

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Dedicated to Professor C.S. Seshadri on his 80 th birthday

Abstract. Let D be a $\mathbb{C}$-derivation on a polynomial ring $\mathbb{C}[x_1, x_2]$. Cerveau [4] asserts that D is surjective as a linear mapping if and only if $D = \frac{\partial}{\partial x_1} + ax_2\frac{\partial}{\partial x_2}$ with respect to a suitable algebraic change of coordinates of $\mathbb{C}[x_1, x_2]$, where $a \in \mathbb{C}$. Inspired by various results in [4], we consider a surjective derivation defined on an affine domain over $\mathbb{C}$ of dimension one or two. Though our proofs are mostly algebraic or algebro-geometric, the idea using a result of Dimca-Saito [5] which is behind the arguments in [4] and based on the differential complex of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is inspiring and affects our arguments.

Introduction

Let $X = \text{Spec } B$ be an affine variety defined over the complex field $\mathbb{C}$ and let $\Omega^1_{X/\mathbb{C}}$ be the sheaf of Kähler differential 1-forms on X. The tangent sheaf $\mathcal{T}_{X/\mathbb{C}}$ is the dual sheaf of $\Omega^1_{X/\mathbb{C}}$, i.e., $\mathcal{T}_{X/\mathbb{C}} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/\mathbb{C}}, \mathcal{O}_X)$ and it is identified with the sheaf of local derivations $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$. A section $\theta \in \Gamma(X, \mathcal{T}_{X/\mathbb{C}})$ is called a *vector field* on X. If X is smooth and has dimension two, θ corresponds to an element of $\Gamma(X, \Omega^1_{X/\mathbb{C}} \otimes \mathcal{K}^{-1}_{X/\mathbb{C}})$, where $\mathcal{K}_{X/\mathbb{C}} = \Omega^2_{X/\mathbb{C}}$. In fact, the wedge product $\wedge : \Omega^1_{X/\mathbb{C}} \times \Omega^1_{X/\mathbb{C}} \to \Omega^2_{X/\mathbb{C}}$ gives rise to an isomorphism of $\mathcal{O}_X$-modules $\Omega^1_{X/\mathbb{C}} \otimes \mathcal{K}^{-1}_{X/\mathbb{C}} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/\mathbb{C}}, \mathcal{O}_X)$. The section θ corresponds to an element $\delta \in \text{Hom}_B(\Omega^1_{B/\mathbb{C}}, B)$ and then to a $\mathbb{C}$-derivation $D \in \text{Der}_{\mathbb{C}}(B, B)$ via $D = \delta \cdot d$, where $d : B \to \Omega^1_{B/\mathbb{C}}$ is the canonical differentiation.

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Let P be a closed point of a smooth affine surface X and let $\{x_1, x_2\}$ be a local system of parameters at P. Then the derivation D is expressed as

$$D = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} \tag{$*$}$$

with $f_1, f_2 \in \mathcal{O}_P$. We can associate a differential 1-form $\omega \in \Omega^1_{X/\mathbb{C},P}$ to D by setting $\omega = -f_2 dx_1 + f_1 dx_2$. Then it is easy to see that

$$\omega \wedge dg = -D(g) dx_1 \wedge dx_2$$

for every $g \in \mathcal{O}_P$. By a change of local coordinates $\{x'_1, x'_2\}$, we have

$$D = f'_1 \frac{\partial}{\partial x'_1} + f'_2 \frac{\partial}{\partial x'_2}, \quad \omega' = -f'_2 dx'_1 + f'_1 dx'_2$$

and $\omega' = J(x'/x)\omega$, where $J(x'/x)$ is the Jacobian determinant of $\{x'_1, x'_2\}$ with respect to $\{x_1, x_2\}$. Hence the correspondence $g \mapsto D(g)$ is independent of the choice of local coordinates. The derivation D is *surjective* if it is surjective as an endomorphism of the set B. If $B = \mathbb{C}[x_1, x_2]$ is a polynomial ring, the derivation D of B is surjective if and only if, for every $f \in B$, there exists an element $g \in B$ such that $\omega \wedge dg = -f dx_1 \wedge dx_2$, where ω is defined on B by $(*)$.

In a very interesting article [4], Cerveau asserts as stated in the abstract that a $\mathbb{C}$-derivation D of a polynomial ring $\mathbb{C}[x_1, x_2]$ is surjective if and only if

$$D = \frac{\partial}{\partial x_1} + a x_2 \frac{\partial}{\partial x_2}$$

after an *algebraic* change of variables, where $a \in \mathbb{C}$. Unfortunately, there exists an error in the proof of [4, Lemma 5.5] which is not corrected for the moment and thereby suspends the validity of the abovementioned result. Furthermore, an analytic theory, e.g., the properties of analytic foliations associated to a differential 1-form is used occasionally in the proof of [4].

Inspired by Cerveau's work and his idea, we intend to generalize the results to surjective derivations D of affine domains B over $\mathbb{C}$ with small dimension. Our trial is successful only in the special cases. An element f of B is called an *integral* element with respect to D if $D(f)$ is divisible by f in B. In Nowicki [2], an integral element for a derivation on a polynomial ring is called a *Darboux polynomial*.

In the present article, we prove the following two results Theorems 1 and 2.

Theorem 1. *Let D be a $\mathbb{C}$-derivation of $\mathbb{C}[x_1, x_2]$. Then $D = \partial/\partial x_1$ with respect to a set of variables $\{x_1, x_2\}$ if and only if D is surjective and* $\operatorname{Ker} D \supsetneqq \mathbb{C}$.

We have only to prove the *if* part because the *only if* part is obvious. Since a locally nilpotent $\mathbb{C}$-derivation on $\mathbb{C}[x_1, x_2]$ is always written as $f(x_2)\partial/\partial x_1$ with respect to a suitable set of variables $\{x_1, x_2\}$, the above result gives a characterization for a given reduced $\mathbb{C}$-derivation on $\mathbb{C}[x_1, x_2]$ to be locally nilpotent, where a derivation is *reduced* if the coefficients $f_1, f_2 \in \mathbb{C}[x_1, x_2]$ have no irreducible common factors when we write

$$D = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}.$$

We prove the above theorem in a generalized setting where $\mathbb{C}[x_1, x_2]$ is replaced by a factorial affine domain B of dimension two over $\mathbb{C}$ such that $B^* = \mathbb{C}^*$. In this case, the reducedness of D is equivalent to the condition that the divisorial part of D on the smooth part of $\operatorname{Spec} B$ is zero (cf. [12]). Thus we obtain a different algebraic characterization of $\mathbb{C}[x_1, x_2]$ in terms of a surjective derivation.

In the subsequent proof, we do not make a full use of the surjectivity of D but only a partial condition derived from the surjectivity of D, that is, the existence of an element $t \in B$ with $D(t) = \delta(dt) = 1$. If R is a normal affine domain of dimension one, we take R as the above B. Then the module of differential forms $\Omega^1_{R/\mathbb{C}}$ is a free R-module Rdt since the R-module homomorphism $\delta : \Omega^1_{R/\mathbb{C}} \to R$ induces an isomorphism, and $\delta(\omega) = g \in R$ for every $\omega \in \Omega_{R/\mathbb{C}}$ if we write $\omega = gdt$. Since D is surjective, $g = D(h)$ for some element $h \in R$. Hence it follows that $\delta(\omega) = D(h) = \delta(dh)$, i.e., $\delta(\omega - dh) = 0$. Since δ is an isomorphism of R-modules, we have $\omega = dh$. Namely, every (closed) form on a curve $\operatorname{Spec} R$ is *exact* in the case of $\dim R = 1$. In view of the isomorphism δ for R, the exactness of any differential 1-form is equivalent to the surjectivity of D. In the case $\dim B \geq 2$, see [4, Proposition 1.2] for the interpretation of D being surjective. We then use the mixed Hodge structure on $\operatorname{Spec} R$ to conclude that $\operatorname{Spec} R$ is in fact the affine line. In the case where $B = \mathbb{C}[x_1, x_2]$, the assumption that $\operatorname{Ker} D \supsetneqq \mathbb{C}$ yields an $\mathbb{A}^1$-fibration $\operatorname{Spec} B \to \operatorname{Spec} A$ with $A = \operatorname{Ker} D$ and consequently gives a proof of the above theorem 1. One can say that the surjectivity of D is *less* algebraic than it appears from the definition. We give several proofs of Theorem 1.2 when $\dim R = 1$ in order to illustrate the condition of surjectivity of D. It is still a mysterious condition when $\dim B > 1$.

Although there is a gap in the proof, the correct parts combined together in [4] imply the following result.

Theorem 2. *Let D be a surjective derivation of $\mathbb{C}[x_1, x_2]$. Then there exists an integral element with respect to D.*

It should be remarked that most $\mathbb{C}$-derivations of $\mathbb{C}[x_1, x_2]$ have no integral elements. In Nowicki [1, 2], a $\mathbb{C}$-derivation without integral elements is called *simple* and many examples are given.

Besides the reference [4], some observations have been made in [3, 18] on surjective derivations on a polynomial ring $R[x, y]$ of dimension two over $R = \mathbb{C}$ or R being a $\mathbb{Q}$-algebra, especially on the relation between surjectivity and local nilpotence of D under the assumption that D has divergence zero. In fact, if D is a surjective derivation of $\mathbb{C}[x, y]$ with divergence zero (see [3, 6] for the definition) then $\mathrm{Ker}\, D \supsetneqq \mathbb{C}$. Hence the above theorem implies that D is locally nilpotent. This is proved in [3] for a $\mathbb{Q}$-algebra R instead of $\mathbb{C}$.

When we looked for examples of derivations without integral elements, Nina Gupta of TIFR kindly found us the references of Nowicki [1, 2]. L. Makar-Lomanov also constructed for us an example (see Example 3.13). We are very thankfulful to both of them.

1. Case of dimension one and related remarks

Let R be a normal affine domain of dimension one. Suppose that R has a surjective $\mathbb{C}$-derivation. Let t be an element of R such that $D(t) = 1$. Let $R_0 = \mathbb{C}[t]$. Then R_0 is a subring of R such that $D(R_0) \subseteq R_0$ and D is locally nilpotent on R_0. The derivation D corresponds to an element $\delta \in \mathrm{Hom}_R(\Omega_{R/\mathbb{C}}, R)$ such that $D(z) = \delta(dz)$ for every $z \in R$, where δ is an R-homomorphism. Since $D(t) = 1$, δ is a surjective homomorphism. Consider an exact sequence of differential R-modules

$$\Omega_{R_0/\mathbb{C}} \otimes_{R_0} R \xrightarrow{\alpha} \Omega_{R/\mathbb{C}} \xrightarrow{\beta} \Omega_{R/R_0} \to 0,$$

where $\Omega_{R_0/\mathbb{C}} \otimes_{R_0} R$ is a free R-module of rank one generated by $dt \otimes 1$ and hence the homomorphism α is injective. Since Spec R is smooth by the hypothesis, $\Omega_{R/\mathbb{C}}$ is a locally free module of rank one and the homomorphism δ is an R-isomorphism since it is surjective. In particular, $\Omega_{R/\mathbb{C}}$ is an R-free module generated by dt. Then $\Omega_{R/R_0} = 0$, which implies that the induced morphism $q : \mathrm{Spec}\, R \to \mathrm{Spec}\, R_0$ is étale.

Lemma 1.1. *The morphism q is surjective. Hence a linear form $t - c$ is not invertible in R for every $c \in \mathbb{C}$.*

Proof. Suppose that q is not surjective and the point $t = 0$ is not in the image of q. Then t is invertible in R. Hence there exists an element u of R such that $D(u) = 1/t$. Write the minimal equation of u over $\mathbb{C}(t)$ in the form

$$a_0(t)u^n + a_1(t)u^{n-1} + \cdots + a_{n-1}(t)u + a_n(t) = 0,$$
$$\forall\ a_i(t) \in R_0,\ \gcd(a_0(t), \ldots, a_n(t)) = 1.$$

If we have another equation of u with degree n

$$b_0(t)u^n + b_1(t)u^{n-1} + \cdots + b_{n-1}(t)u + b_n(t) = 0,\ \forall\ b_i(t) \in R_0,$$

the condition $\gcd(a_0(t), \ldots, a_n(t)) = 1$ implies $b_i(t)/a_i(t)$ is an element of R_0 which is independent of i. Applying D to the first equation, we have

$$ta_0'(t)u^n + (ta_1'(t) + na_0(t))\, u^{n-1} + \cdots$$
$$+ (ta_i'(t) + (n-i+1)a_{i-1}(t))\, u^{n-i} + \cdots + (ta_n'(t) + a_{n-1}(t)) = 0.$$

Hence we have

$$\frac{ta_0'(t)}{a_0(t)} = \frac{ta_1'(t) + na_0(t)}{a_1(t)} = c \in \mathbb{C}.$$

Let $r = \deg a_0(t)$. Then the equality $ta_0'(t) = ca_0(t)$ implies that $c = r$ and $a_0(t) = c_0 t^r$ with $c_0 \in \mathbb{C}^*$. We may assume that $c_0 = 1$. Write

$$a_1(t) = d_0 t^s + d_1 t^{s-1} + \cdots + d_{s-1}t + d_s, \quad d_0 \neq 0.$$

Then the equation $ta_1'(t) + na_0(t) = ca_1(t)$ is written as

$$sd_0t^s + (s-1)d_1t^{s-1} + \cdots + d_{s-1}t + nt^r = rd_0t^s + rd_1t^{s-1} + \cdots + rd_{s-1}t + rd_s.$$

This implies that $s = r$ and hence $n = 0$. This is a contradiction. □

The following result determines the structure of R.

Theorem 1.2. *Let R be a normal affine domain of dimension* 1 *over $\mathbb{C}$. Assume that R has a surjective $\mathbb{C}$-derivation D. Then the above morphism q is an isomorphism. Hence $R = \mathbb{C}[t]$ and $D = d/dt$.*

Proof. Let $X = \operatorname{Spec} R$, let C be a smooth completion of X and let $\Delta = C \setminus X$, which we regard as a reduced effective divisor on C. The mixed Hodge structure theorem due to Deligne (see [19, Chapter 8]) gives the following isomorphism

$$H^1(X; \mathbb{C}) \cong H^0(C, K_C + \Delta) \oplus H^1(C, \mathcal{O}_C),$$

where we note that $\Omega^1_{C/\mathbb{C}}(\log \Delta) = \mathcal{O}_C(K_C + \Delta)$. By the Riemann-Roch theorem, we have

$$h^0(C, K_C + \Delta) = g - 1 + n,\ h^1(C, \mathcal{O}_C) = g,\ n = \deg \Delta.$$

Hence $\dim H^1(X;\mathbb{C}) = 2g-1+n$. The de Rham theorem gives a non-degenerate pairing

$$H_1(X;\mathbb{C}) \times H^1(X;\mathbb{C}) \to \mathbb{C}, \quad (\gamma,\omega) \mapsto \int_\gamma \omega, \tag{1}$$

where γ is a closed loop in X and ω is a differential 1-form having at most logarithmic singularities along Δ. Then ω restricted onto X is a regular form, and hence $\omega|_X = dh$ for an element $h \in R$. This follows from the surjectivity of D. Since γ is a loop in X, we have

$$\int_\gamma \omega = \int_\gamma \omega|_X = \int_\gamma dh = 0.$$

The last equality follows from Stokes' theorem. Since the pairing (1) is non-degenerate, this implies that $H^1(X;\mathbb{C}) = 0$. Namely, $2g-1+n=0$. Since $n>0$, this is equivalent to $g=0$ and $n=1$. Hence X is isomorphic to the affine line.

Now write $R = \mathbb{C}[z]$. Then $D = d/dz$ possibly after a suitable change of z. Since $D(t) = 1$, we have $t = z+c$ with $c \in \mathbb{C}$. Hence $R = \mathbb{C}[t]$. □

The second proof of Theorem 1.2. In the above proof, one can use the algebraic de Rham theorem by Grothendieck [8]. In fact, if X is a smooth affine variety, it is shown that the following equality holds

$$H^*(X;\mathbb{C}) = H^*(X, \Gamma(X, \Omega^*_{X/\mathbb{C}})).$$

Hence if $\dim X = 1$ and every regular 1-form on X is exact, it follows that $H^1(X;\mathbb{C}) = 0$, which implies that $X \cong \mathbb{A}^1$. □

The third proof of Theorem 1.2. We can give a different proof without using the de Rham theorem. In the proof of Lemma 1.1, it is shown that the morphism $q : \text{Spec } R \to \text{Spec } R_0$ is surjective and étale. Let $X = \operatorname{Spec} R$ and $Z = \operatorname{Spec} R_0$. Since Z is simply connected as $Z \cong \mathbb{A}^1$, q is an isomorphism if we show that q is finite. This follows from the following lemma. In fact, let $\widetilde{R}$ be the normalization of R_0 in R and let $\widetilde{X} = \text{Spec } \widetilde{R}$. Then X is an open set of $\widetilde{X}$. If $X \subsetneqq \widetilde{X}$, then R cannot have a surjective derivation. Hence $X = \widetilde{X}$ and q is finite.

Lemma 1.3. *Let C be a smooth affine curve and let P be a point of C. Then there exists a regular 1-form on C which is not exact on $C \setminus \{P\}$.*

Proof. Let $\Omega^1_{C/\mathbb{C}}(P)$ be the sheaf of 1-forms with at most a simple pole at P. Then $\Omega^1_{C/\mathbb{C}}(P)$ is a coherent $\mathcal{O}_C$-Module containing properly

$\Omega^1_{C/\mathbb{C}}$ as an $\mathcal{O}_C$-Module. Since C is affine, we have a proper containment

$$\Gamma(C, \Omega^1_{C/\mathbb{C}}(P)) \supsetneqq \Gamma(C, \Omega^1_{C/\mathbb{C}}).$$

Hence there exists a 1-form η on C which is regular on $C \setminus \{P\}$ and has a simple pole at P. Suppose that η is exact, i.e., $\eta = dh$ for a rational function h. Then h must be regular. In fact, if h has a pole of order n at P, then dh has a pole of order $n+1$. Hence η cannot have a simple pole at P. But this contradicts the choice of η. □

Employing the argument in the proof of Lemma 1.3, we can show the following result.

Lemma 1.4. *Let $f : X \to Z$ be a non-constant morphism between smooth affine curves. Let η be a regular 1-form on Z. If $f^*(\eta)$ is exact on X, then η is exact on Z.*

Proof. Let $\widetilde{f} : \widetilde{X} \to Z$ be the normalization of Z in the function field of X. Then X is an open subset of $\widetilde{X}$. By the assumption, $f^*(\eta) = dh$ for a regular function h on X. The proof of Lemma 1.3 shows that it is exact on $\widetilde{X}$. In fact, if h has a pole of order n at a point P of $\widetilde{X} \setminus X$, then dh has a pole of order $n+1$ at P which contradicts the regularity of $f^*(\eta)$. Let W be the normalization of Z in the smallest Galois extension of the function field of Z containing the function field of $\widetilde{X}$. Then the normalization morphism $\nu : W \to Z$ splits as $\nu = \widetilde{f} \circ \mu$, where $\mu : W \to \widetilde{X}$ is also a Galois extension. Let G be the Galois group of the extension $\mathbb{C}(W)/\mathbb{C}(Z)$. Then $\nu^*(\eta) = \mu^*(\widetilde{f}^*(\eta)) = \mu^*(dh) = d(\mu^*(h))$. Let $g \in G$ and let $\alpha(g) : W \to W$ be the automorphism induced by the field automorphism $g : \mathbb{C}(W) \to \mathbb{C}(W)$. Then we have

$$\alpha(g)^*(\nu^*(\eta)) = d(g(\mu^*(h))),$$

where $\alpha(g)^*(\nu^*(\eta)) = \nu^*(\eta)$ for every $g \in G$. Hence by the averaging trick, we have

$$\eta = \frac{1}{|G|} \sum_{g \in G} d(g(\mu^*(h))) = d\left(\frac{1}{|G|} \sum_{g \in G} g(\mu^*(h)) \right),$$

where $\frac{1}{|G|} \sum_{g \in G} g(\mu^*(h))$ is a regular function on Z. So, η is exact on Z. □

Lemma 1.4 gives a different proof of Lemma 1.1.

The second proof of Lemma 1.1 With the notations in Lemma 1.1, suppose that $q : X \to \mathbb{A}^1$ is not surjective. By Lemma 1.3, we can find a regular 1-form η on $q(X)$ which is not exact on $q(X)$. By Lemma

1.4, $q^*(\eta)$ is not exact on X. This is a contradiction since every regular 1-form on X is exact by the surjectivity of D. $\square$

To understand what the surjectivity of D implies, we conduct an elementary calculation to verify Theorem 1.2 in the case where R is rational over $\mathbb{C}$.

Lemma 1.5. *With the above notations, suppose that R is rational and D is surjective on R. Then $R = \mathbb{C}[t]$ and $D = d/dt$.*

Proof. We can write

$$R = \mathbb{C}\left[t, \frac{1}{\prod_{i=1}^r (t-\alpha_i)}\right], \quad \alpha_i \in \mathbb{C},\ \alpha_i \neq \alpha_j\ (i \neq j).$$

We only consider the case where the derivation D is written as

$$D = \frac{f(t)}{\prod_{i=1}^r (t-\alpha_i)^{n_i}} \frac{d}{dt}, \quad f(t) \in \mathbb{C}[t],\ f(\alpha_i) \neq 0\ (\forall\ i),\ n_i > 0.$$

Since D is surjective, there exists an element $u \in R$ such that $D(u) = 1$. Write

$$u = \frac{g(t)}{\prod_{i=1}^r (t-\alpha_i)^{m_i}}, \quad g(\alpha_i) \neq 0\ (\forall\ i),\ m_i \in \mathbb{Z}.$$

The equation $D(u) = 1$ yields

$$\frac{f(t)}{\prod_{i=1}^r (t-\alpha_i)^{n_i}} \cdot \left\{ \frac{g'(t)}{\prod_{i=1}^r (t-\alpha_i)^{m_i}} - \frac{g(t)}{\prod_{i=1}^r (t-\alpha_i)^{m_i}} \cdot \sum_{i=1}^r \frac{m_i}{(t-\alpha_i)} \right\} = 1.$$

This is written as follows

$$f(t) \left\{ g'(t) \prod_{i=1}^r (t-\alpha_i) - g(t) \sum_{i=1}^r m_i \cdot \frac{\prod_{i=1}^r (t-\alpha_i)}{t-\alpha_i} \right\} = \prod_{i=1}^r (t-\alpha_i)^{n_i+m_i+1}.$$

Since $f(\alpha_i) \neq 0$ for all i, it follows that $n_i + m_i + 1 = 0$ for all i and hence $f(t) \in \mathbb{C}^*$. After changing $g(t)$ by $f(0)g(t)$, we may assume that $f(t) = 1$. Then $\deg g(t) = s > 0$, and $g(t) = 0$ has no multiple roots. We can write

$$g(t) = c \prod_{j=1}^s (t-\beta_j),\ \beta_j \neq \beta_\ell\ (j \neq \ell),\ \beta_j \neq \alpha_i.$$

The above equation is then written as

$$\sum_{j=1}^s \frac{c}{t-\beta_j} - \sum_{i=1}^r \frac{cm_i}{t-\alpha_i} = \frac{1}{\prod_{i=1}^r (t-\alpha_i) \prod_{j=1}^s (t-\beta_j)},$$

where $-m_i = n_i + 1 \geq 2$. Assume that $r \geq 1$. Then comparison of the top t-degrees of both sides of this equation after multiplied $\prod_{i=1}^{r}(t - \alpha_i)\prod_{j=1}^{s}(t - \beta_j)$ yields an equality

$$1 - \sum_{i=1}^{r} m_i = 0.$$

But this is a contradiction. If $r = 0$, then $R = \mathbb{C}[t]$. □

The use of Zariski's lemma [21, Lemma 4] enables us to drop in the statement of Theorem 1.2 the assumption that R is normal.

Theorem 1.6. *Let $(R, \mathfrak{m})$ be a geometric local ring* [1] *and let D be a $\mathbb{C}$-derivation of R. Assume that there exists an element $t \in R$ with $D(t) = 1$. Then the following assertions hold.*

(1) *If* $\dim R = 1$ *then R is normal.*
(2) *Assume that* $\dim R = 2$ *and R is normal. Then R is regular.*

Proof. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then the derivation D extends to a derivation on $\widehat{R}$ in a natural fashion, which we denote by $\widehat{D}$. Then we have $\widehat{D}(t) = 1$. By Zariski's lemma, $\widehat{R} = \mathfrak{o}[[\, t\,]]$ for a complete local ring $\mathfrak{o}$ such that the restriction of $\widehat{D}$ onto $\mathfrak{o}$ is zero and t is analytically independent over $\mathfrak{o}$. Note that $\widehat{R}$ is reduced by Nagata [15, Theorem (37.5)]. If $\dim R = 1$ then $\dim \mathfrak{o} = 0$ and hence $\mathfrak{o} = \mathbb{C}$. So, $\widehat{R} = \mathbb{C}[[\, t\,]]$, whence R is normal because $\widehat{R}$ is a faithfully flat R-module. If $\dim R = 2$ and R is normal, then $\widehat{R}$ is normal by [15, Theorem (37.5)]. Hence $\mathfrak{o}$ is a complete normal ring of dimension one. Then $\mathfrak{o} = \mathbb{C}[[\, u\,]]$ and $\widehat{R} = \mathbb{C}[[t, u]]$, whence R is regular by the faithfully flat descent of regularity. □

Note that the hypothesis that D is surjective is stronger than the hypothesis that there exists an element $t \in R$ with $D(t) = 1$. If we use the stronger hypothesis, we can give a different proof for R being normal if $\dim R = 1$.

Proposition 1.7. *Let R be an affine domain of dimension one defined over $\mathbb{C}$. Assume that R has a surjective $\mathbb{C}$-derivation D. Then R is normal.*

Proof. Suppose that R is not normal. Let $\widetilde{R}$ be the normalization of R. Then, by [17], the derivation D extends to a $\mathbb{C}$-derivation $\widetilde{D}$ of $\widetilde{R}$. Let $\mathfrak{c}$ be the conductor of R in $\widetilde{R}$. Then $\mathfrak{c}$ is an ideal of R and $\widetilde{R}$ as well. Furthermore, the closed set $V(\mathfrak{c})$ is the singular locus of Spec R

[1] Namely, R is the local ring of a closed point of an algebraic variety.

and hence $R/\mathfrak{c}$ is a $\mathbb{C}$-vector space of finite dimension. The conductor $\mathfrak{c}$ is defined as the set

$$\mathfrak{c} = \{b \in \widetilde{R} \mid bx \in R, \ \forall\, x \in \widetilde{R}\}.$$

Let $b \in \mathfrak{c}$. Then, for every $x \in \widetilde{R}$, we have $D(bx) = D(b)x + b\widetilde{D}(x)$, whence $D(b)x \in R$ as $D(bx), b\widetilde{D}(x) \in R$. So, $D(\mathfrak{c}) \subseteq \mathfrak{c}$. Hence D induces a $\mathbb{C}$-linear endomorphism $\overline{D}$ of $R/\mathfrak{c}$. We have the following commutative exact diagram of $\mathbb{C}$-modules

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{c} & \to & R & \to & R/\mathfrak{c} & \to & 0 \\ & & \big\downarrow D_c & & \big\downarrow D & & \big\downarrow \overline{D} & & \\ 0 & \to & \mathfrak{c} & \to & R & \to & R/\mathfrak{c} & \to & 0 \end{array}$$

where the middle vertical arrow D is a surjection and the right vertical arrow $\overline{D}$ is thereby a surjective $\mathbb{C}$-linear endomorphism of the finite-dimensional $\mathbb{C}$-vector space $R/\mathfrak{c}$. Hence $\overline{D}$ is a bijection. Then, by the snake lemma, $\operatorname{Ker} D_c = \operatorname{Ker} D$, and $\operatorname{Ker} D = \mathbb{C}$ for D is a non-zero $\mathbb{C}$-derivation of R and $\dim R = 1$. This is a contradiction because $\mathfrak{c} \cap \mathbb{C} = (0)$. This shows that R is normal. □

The surjectivity of D does not necessarily imply the normality in the higher-dimensional case.

Example 1.8. Let $B = \mathbb{C}[X, Y, Z]/(X^2 + Y^3)$ and let x, y, z be respectively the residue classes of X, Y, Z in B. Define a derivation $\mathcal{D}$ on $\mathbb{C}[X, Y, Z]$ by

$$\mathcal{D} = \frac{1}{2}X\frac{\partial}{\partial X} + \frac{1}{3}Y\frac{\partial}{\partial Y} + \frac{\partial}{\partial Z}.$$

Then $\mathcal{D}$ maps the ideal $(X^2 + Y^3)$ into itself. Hence $\mathcal{D}$ induces a $\mathbb{C}$-derivation D on the ring B. Since $\mathcal{D}$ is easily verified to be surjective (cf. Lemma 1.11 below), the derivation D is surjective as well. The ring B is not normal.

In the case of an affine plane curve, we have the following result.

Proposition 1.9. *Let $R := \mathbb{C}[X, Y]/(F)$ be the coordinate ring of an irreducible, affine plane curve $C := \{F(X, Y) = 0\}$. Let x, y be the residue classes of X, Y in R respectively. Write the residue classes of the partial derivatives F_X, F_Y by f_x, f_y respectively. Then the following assertions hold.*

(1) *There exists a $\mathbb{C}$-derivation D of R with $D(x) = 1$ if and only if $f_x \in f_y R$. If this condition is satisfied, then the curve C is smooth.*

(2) *Suppose that there exists a* $\mathbb{C}$*-derivation* D *of* R *with* $D(x) = 1$. *Then either* R *is a polynomial ring* $\mathbb{C}[x]$ *or the element* f_y *is a non-constant, invertible element in* R.

Proof. (1) Let Δ be a $\mathbb{C}$-derivation on $\mathbb{C}[X, Y]$. Then $\Delta = A(\partial/\partial X) + B(\partial/\partial Y)$ with $A, B \in \mathbb{C}[X, Y]$, and Δ induces a $\mathbb{C}$-derivation D on R if and only if $\Delta(F) \in (F)$. Further, if D is the induced derivation, then we have $f_x D(x) + f_y D(y) = 0$. If $D(x) = 1$, then $f_x = -f_y D(y) \in f_y R$ as $D(y) \in R$. Conversely, suppose that $f_x = -f_y g$ with $g \in R$. Then there exist elements $G, H \in \mathbb{C}[X, Y]$ such that g is the residue class of G and $F_X = -F_Y G + FH$. Set $\Delta = (\partial/\partial X) + G(\partial/\partial Y)$. Then $\Delta(F) = F_X + F_Y G = FH$. Hence Δ induces a $\mathbb{C}$-derivation D with $D(x) = 1$. By Theorem 1.6, R is normal and hence the curve C is smooth if this condition is satisfied.

(2) Since C is smooth, the ideal (f_x, f_y) is a unit ideal. Suppose that f_y is a constant $c \in \mathbb{C}$. Then $F_Y - c$ is divisible by F. Since $\deg_Y F_Y < \deg_Y F$, it follows that $F_Y = c$. This implies that $F = cY - G(x)$. So, $R = \mathbb{C}[x]$. Suppose that f_y is not a constant. Since $(f_x, f_y) = R$, we find elements $a, b \in R$ such that $af_x + bf_y = 1$ in R. Hence $af_x/f_y + b = 1/f_y$. By the assertion (1), $f_x/f_y \in R$ and hence $1/f_y \in R$. □

By the following remark, there exist an affine normal domain R over $\mathbb{C}$ with genus $g > 0$ and a $\mathbb{C}$-derivation D such that $D(x) = 1$ for some element $x \in R$.

Remark 1.10. Let C be a smooth projective curve of genus g. By a classical result of F. Severi (cf. [7, Proposition 8.1]), there is a morphism $f : C \to \mathbb{P}^1$ of degree $n \geq g + 1$ such that, over every point $P \in \mathbb{P}^1$, there is at most one point, say $\widetilde{P}$ (if it exists) which is ramified over P and has ramification index 2. Fix a point $P_\infty \in \mathbb{P}^1$. Let C_0 be the open set obtained from C by omitting all the points lying over P_∞ and also all the ramified points. Let R be the coordinate ring of the affine curve C_0. The induced morphism $f : C_0 \to \mathbb{A}^1$ is surjective and étale. Let $\mathbb{C}[x]$ be the coordinate ring of $\mathbb{A}^1$. Since dx is a nowhere vanishing regular 1-form on C_0, we have $\Omega_{R/\mathbb{C}} = Rdx$. In fact, for every element $a \in R$, the ratio da/dx, which lies in the function field of R, has non-negative value at the discrete valuation ring corresponding to any point of C_0, i.e., $da/dx \in R$. This implies that $D := d/dx$ is a $\mathbb{C}$-derivation of R and that $D(x) = 1$. We have seen in Theorem 1.2 that if $g > 0$ then D cannot be surjective. Furthermore, since $\Omega^1_{R/\mathbb{C}} = Rdx$, R is a complete intersection by a result of Murthy-Towber [14, 9]. Hence there are many examples of normal 1-dimensional affine domains with a $\mathbb{C}$-derivation D such that $D(x) = 1$ for some $x \in R$ but D is not surjective. $\square$

The following result gives a construction of a surjective derivation. A $\mathbb{C}$-derivation D on a $\mathbb{C}$-algebra R is called *locally finite* if, for every element $a \in R$, the $\mathbb{C}$-vector space spanned by the set $\{D^i(a) \mid i \geq 0\}$ has finite dimension.

Lemma 1.11. *Let R be a $\mathbb{C}$-algebra and let D be a $\mathbb{C}$-derivation on R. Let t be an indeterminate and let $B = R[t]$. Define a $\mathbb{C}$-derivation $\widetilde{D}$ on B by $\widetilde{D}(a) = D(a)$ for every $a \in R$ and $\widetilde{D}(t) = 1$. Then $\widetilde{D}$ is surjective if D satisfies one of the conditions:*

(1) *$D = 0$.*
(2) *D is locally nilpotent.*
(3) *D is locally finite.*
(4) *$R \setminus \mathbb{C} \subseteq \operatorname{Im} D$.*

Proof. Since we have implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3), we prove the assertion in the cases (3) and (4).

(3) Let a be an element of R. It suffices to show that $at^n \in \operatorname{Im} \widetilde{D}$ for every $n \geq 0$. If $D(a) = 0$ then $at^n = \widetilde{D}(at^{n+1})/(n+1) \in \operatorname{Im} \widetilde{D}$. Suppose that $D^m(a) = 0$. By induction on m, we show that $at^n \in \operatorname{Im} \widetilde{D}$ for every $n \geq 0$. In fact, we have

$$\widetilde{D}(at^{n+1}) = (n+1)at^n + D(a)t^{n+1},$$

where $D(a)t^{n+1} \in \operatorname{Im}\widetilde{D}$ by the induction hypothesis because $D^{m-1}(D(a)) = 0$. Hence $at^n \in \operatorname{Im} \widetilde{D}$. Meanwhile, since D is locally finite, there exists a relation

$$c_0 a + c_1 D(a) + \cdots + c_p D^p(a) = 0, \quad \forall\ c_i \in \mathbb{C}.$$

If $c_0 \neq 0$, then $a \in \operatorname{Im} D$. If $c_0 = \cdots = c_{\ell-1} = 0$ and $c_\ell \neq 0$, then $a - a_0 \in \operatorname{Im} D$ for an element $a_0 \in R$ with $D^\ell(a_0) = 0$. In fact, it suffices to put

$$a_0 = a + \frac{1}{c_\ell}\bigl(c_{\ell+1} D(a) + \cdots + c_p D^{p-\ell}(a)\bigr).$$

Since $a_0 t^n \in \operatorname{Im} \widetilde{D}$ for $\forall\ n \geq 0$ by the above observation, we may assume $a \in \operatorname{Im} D$ in order to show that $at^n \in \operatorname{Im} \widetilde{D}$. Write $a = D(a')$ with $a' \in R$. Since

$$at^n = D(a')t^n = \widetilde{D}(a't^n) - na't^{n-1},$$

we are done by induction on n.

(4) Note that $ct^n \in \operatorname{Im} \widetilde{D}$ for $c \in \mathbb{C}$ and $n \geq 0$, for $\widetilde{D}(ct^{n+1}/(n+1)) = ct^n$. If $a \in R \setminus \mathbb{C}$, we show by induction on n that $at^n \in \operatorname{Im} \widetilde{D}$. If $n = 0$ then $a \in \operatorname{Im} D$ by the assumption. Write $a = D(a')$ with $a' \in R$. Then $at^n = D(a')t^n = D(a't^n) - na't^{n-1}$ and $a't^{n-1} \in \operatorname{Im} \widetilde{D}$ by the induction hypothesis, whence $at^n \in \operatorname{Im} \widetilde{D}$. □

2. Proof of Theorem 1

Let B be a factorial affine domain of dimension two over $\mathbb{C}$ such that $B^* = \mathbb{C}^*$. Throughout the present section, the domain B satisfies these conditions. We assume that B is endowed with a $\mathbb{C}$-derivation D such that $\operatorname{Ker} D \supsetneqq \mathbb{C}$. Wherever we need the surjectivity condition on D, we mention it explicitly.

Lemma 2.1. *Let D be a non-zero $\mathbb{C}$-derivation on B such that* $\operatorname{Ker} D \supsetneqq \mathbb{C}$. *Let $A = \operatorname{Ker} D$. Then the following assertions hold.*

(1) *A is a polynomial ring in one variable.*
(2) *$Q(A)$ is algebraically closed in $Q(B)$, where $Q(A)$ and $Q(B)$ are the quotient fields of A and B respectively.*

Proof. (1) The ring A is integrally closed in B. In fact, if z is an element of B which is integral over A, let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \quad \forall a_i \in A$$

be a monic equation of z over A. We may assume that the degree n is minimal among such monic equations for z. Applying D to $f(z)$, we obtain a relation

$$\frac{1}{n} f'(z) D(z) = 0,$$

where $\frac{1}{n} f'(z)$ is monic in z and hence nonzero by the hypothesis. Then $D(z) = 0$. So, $z \in A$. It is clear that $\dim A = 1$ as D is non-trivial and $A = B \cap \operatorname{Ker} \widetilde{D}$, where $\widetilde{D}$ is the $\mathbb{C}$-derivation of $Q(B)$ which is the natural extension of D. Hence A is finitely generated by a lemma of Zariski [20]. Furthermore, $Q(A)$ is rational over $\mathbb{C}$ because B is factorial. Since $A^* = \mathbb{C}^*$ as $B^* = \mathbb{C}^*$, it follows that A is a polynomial ring $\mathbb{C}[\xi]$ in one variable ξ.

(2) Suppose that z is an element of $Q(B)$ which is algebraic over $Q(A)$. Then there is an algebraic relation

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \ \forall a_i \in A, \ a_0 \neq 0.$$

Then the element $a_0 z$ is integral over A. Hence $a_0 z \in A$ by the assertion (1). So, $z \in Q(A)$ and $Q(A)$ is algebraically closed in $Q(B)$. □

Let $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and $p : Y \to X$ the morphism associated to the natural inclusion $A \hookrightarrow B$. Since $Q(A)$ is algebraically closed in $Q(B)$, the general fibers of p are smooth curves. The derivation D induces a derivation D_c on the fiber $F_c = \operatorname{Spec} B/(\xi - c)$, which is surjective provided so is D.

We consider, in general, a fibration $p : Y \to X$, which is by definition a dominant morphism of algebraic varieties Y, X with smooth irreducible general fibers. A fiber F of p is called *singular* (resp. *reducible*) if F is scheme-theoretically not isomorphic to a general fiber (resp. if F has more than one irreducible components). If F is reducible, any irreducible component of F is called a *fiber component* of F. The fibration p is called *completely separated* if every reducible fiber is a disjoint union of irreducible fiber components. If the morphism $p : Y \to X$ is smooth, then it is completely separated.

For a $\mathbb{C}$-algebra B and a $\mathbb{C}$-derivation D of B, we say that an ideal I of B is *D-stable* if $D(I) \subseteq I$. Similarly, a closed set V of $\operatorname{Spec} B$ is *D-stable* if the radical defining ideal of V in B is D-stable. We resume our assumptions and notations. We give a result concerning D-stability of the irrdecible components of a fiber F_c of p. Let F_c be the fiber of p defined by $\xi - c = 0$. Let $R = B/(\xi - c)$. Write $\xi - c = f_1^{\alpha_1} \cdots f_n^{\alpha_n}$ be the prime decomposition of $\xi - c$ in B with $\forall \alpha_i > 0$.

Lemma 2.2. *With the above notations, we have the following assertions.*

(1) *Let $\mathfrak{p}$ be a minimal prime divisor of (0) in R. Then $\mathfrak{p}$ is D-stable, and hence $R/\mathfrak{p}$, which is equal to B/f_i for some i, has the $\mathbb{C}$-derivation induced by D. In particular, the nilradical $\sqrt{0}$ is D-stable.*

(2) *Every irreducible fiber F_c is reduced. It is also smooth if D is surjective and B is smooth.*

(3) *Assume that D is surjective. Let $F_c = \operatorname{Spec} B/(\xi - c)$ be an irreducible fiber. Then the $\mathbb{C}$-derivation D_c induced on $B/(\xi - c)$ is surjective.*

Proof. (1) The result is known (see e.g., [6]). But we give a proof for the completeness of the proof. Let $\mathfrak{p}$ be a minimal prime divisor of (0) in R. Then $\mathfrak{p}$ is the ideal quotient $(0 : a)$ for an element $a \in R$. Namely, $\mathfrak{p} = \{z \in R \mid az = 0\}$. Note that $(0 : a) = (0 : a^2)$ since $\mathfrak{p}$ is a prime ideal. Applying D to $az = 0$, we have $aD(z) + D(a)z = 0$, whence $a^2 D(z) = 0$ and $D(z) \in (0 : a^2) = \mathfrak{p}$. If $a \notin \mathfrak{p}$, then $(0 : a) = (0 : a^2)$ holds. But if $a \in \mathfrak{p}$ then $a^2 = 0$ and $(0 : a^2) = R$. Hence this argument breaks. We need another argument to show that $\mathfrak{p}$ is D-stable. First, let (R, M) be an Artin local ring over $\mathbb{C}$ with a $\mathbb{C}$-derivation D. Then the maximal ideal M is D-stable. In fact, if b is an element of M with $D(b) \notin M$. Then $b^n = 0$ for some $n > 0$ and $D(b)$ is a unit. We have $0 = D(b^n) = nb^{n-1}D(b)$, whence $b^{n-1} = 0$. Repeating this argument, we have $b = 0$, which is a contradiction. Now let P be a minimal prime divisor of (0) in a Noetherian ring R over $\mathbb{C}$. Consider the local ring (R_P, PR_P). Then it is an Artin local ring defined over $\mathbb{C}$ with a $\mathbb{C}$-derivation D. By the foregoing argument, PR_P is D-stable. Let b be an element of P. Then $sD(b) \in P$ for some element $s \notin P$. Then $D(b) \in P$. Hence P is D-stable. Let I be a D-stable ideal and let P be a minimal prime divisor of R. We can argue with the residue ring R/I and the induced derivation $\overline{D}$. Then P/I is a minimal prime divisor of (0) in R/I. Hence it follws that P is D-stable. Now we know that any minimal prime ideal $\mathfrak{p}$ of (0) is D-stable and D induces a $\mathbb{C}$-derivation on the residue ring $R/\mathfrak{p}$. Write $\sqrt{0} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_n$ with minimal prime divisors $\mathfrak{p}_i$ of (0) in R. Since each $\mathfrak{p}_i$ is D-stable, $\sqrt{0}$ is also D-stable.

(2) We show that if $F_c = \operatorname{Spec} R$ is irreducible, it is reduced. In fact, suppose that $\operatorname{Spec} R$ is not reduced. Then $\xi - c = f^n$ with $n > 1$. Then $0 = D(\xi - c) = nf^{n-1}D(f)$, whence $D(f) = 0$ and $f \in \mathbb{C}[\xi]$. Writing f as an element in $\mathbb{C}[\xi]$ and substituting it in $\xi - c = f^n$, we obtain an algebraic relation of ξ over $\mathbb{C}$. This is a contradiction. Hence $\operatorname{Spec} R$ is reduced. Let P be a singular point of F_c and let $\{x, y\}$ be a local system of parameters of $Y = \operatorname{Spec} B$ at the point P. Let $f = \xi - c$. Let f_x, f_y be the partial derivatives of f with respect to x, y in the local ring $\mathcal{O}_{Y,P}$ and write $f_x = \overline{f}_x h, f_y = \overline{f}_y h$ with $h = \gcd(f_x, f_y)$ in $\mathcal{O}_{Y,P}$. Since F_c is the curve defined locally by $f = 0$, we have $f_x dx + f_y dy = 0$ on F_c. Then $\overline{f}_x(P) = \overline{f}_y(P) = 0$ as P is a singular point of F_c. On the other hand, we have $f_x D(x) + f_y D(y) = 0$ in B because $D(f) = D(\xi - c) = 0$. Hence $D(x) = \overline{f}_y h_1$ and $D(y) = -\overline{f}_x h_1$ with $h_1 \in \mathcal{O}_{Y,P}$. Now the surjectivity of D implies that there exists an element $t \in B$ such that $D(t) = 1$. Then $D(t) = t_x D(x) + t_y D(y) = t_x \overline{f}_y h_1 - t_y \overline{f}_x h_1$ and $D(t)(P) = 0$. This is a contradiction. Hence F_c is smooth.

(3) Let $\rho_c : B \to B/(\xi - c)$ be the residue homomorphism. Then $\rho_c \cdot D = D_c \cdot \rho_c$. Hence D_c is surjective provided so is D. $\square$

Now we can prove the following result which includes Theorem 1 in the introduction.

Theorem 2.3. *Let B be a factorial affine domain of dimension two over $\mathbb{C}$. Assume that $B^* = \mathbb{C}^*$ and B has a surjective $\mathbb{C}$-derivation D with* $\operatorname{Ker} D \supsetneqq \mathbb{C}$. *Then B is isomorphic to a polynomial ring $\mathbb{C}[x, y]$ and D corresponds to the partial derivation $\partial/\partial x$.*

Proof. Let $A = \mathrm{Ker}\, D$. By Lemma 2.1, A is a polynomial ring $\mathbb{C}[\xi]$. Let $Y = \mathrm{Spec}\, B$, $X = \mathrm{Spec}\, A$ and $p : Y \to X$ be the morphism induced by the natural inclusion $A \hookrightarrow B$. By Lemma 2.1, (2), p is the fibration. Let F_c be a general smooth fiber and let $R = B/(\xi - c)$. By Lemma 2.2, there is a surjective $\mathbb{C}$-derivation on R induced by D, which we denote by the same symbol D. By Theorem 1.2, the fiber F_c is isomorphic to the affine line. Hence p is an $\mathbb{A}^1$-fibration over $X = \mathrm{Spec}\, A \cong \mathbb{A}^1$. In fact, if we choose an element $t \in B$ so that $D(t) = 1$, then $R = \mathbb{C}[\,\bar{t}\,]$, where $\bar{t}$ is the residue class of t in R. Since B is factorial and $B^* = \mathbb{C}^*$, by an algebraic characterization of the affine plane [13], Y is isomorphic to $\mathbb{A}^2$ and B is isomorphic to $\mathbb{C}[\xi, \tau]$, where we can take the generator ξ of A as one of the parameters of B. Let $B_0 = \mathbb{C}[\xi, t]$, where $D(t) = 1$, and let $Z = \mathrm{Spec}\, B_0$. Then the inclusion $B_0 \hookrightarrow B$ induces a dominant X-morphism $\pi : Y \to Z$. Since $B_0/(\xi - c) = \mathbb{C}[\,\bar{t}\,] = B/(\xi - c)$ with the second equality being proved in the proof of Theorem 1.2, the morphism π is a bijection. Hence π is an isomorphism by Zariski's main theorem and thereby $B = \mathbb{C}[\xi, t]$. If we make a change of variables $\xi = y$ and $t = x$, then D is identified with $\partial/\partial x$. □

The following result gives a characterization of a locally nilpotent derivation on $\mathbb{C}[x, y]$.

Corollary 2.4. *Let D be a non-trivial reduced $\mathbb{C}$-derivation of $\mathbb{C}[x, y]$. Then D is locally nilpotent if and only if D is surjective and* $\mathrm{Ker} D \supsetneqq \mathbb{C}$.

The following remark is also interesting.

Remark 2.5. Let $B = \mathbb{C}[x, y]$ be the ring of polynomials in two variables and let $\{f, g\}$ be a Jacobian pair, i.e., the Jacobian determinant $\det \partial(f, g)/\partial(x, y) = 1$. Let D be a $\mathbb{C}$-derivation

$$D = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}.$$

Then the Jacobian conjecture holds if and only if D is surjective.

In fact, $D(f) = 0$ and $D(g) = 1$. If D is surjective, then Corollary 2.4 implies that D is locally nilpotent. It is well-known that the polynomial endomorphism of $\mathbb{A}^2$ defined by the pair $\{f, g\}$ is an automorphism if D is locally nilpotent. The converse is also clear. For these facts, we refer to [6, Chapter 4].

3. Some observations in general cases

Let $Y = \mathrm{Spec}\, B$ be a *smooth* affine variety and let D be a surjective $\mathbb{C}$-derivation of B. We summarize the results obtained in this setting.

Lemma 3.1. *The following assertions hold.*

(1) *Let* $\mathfrak{p}$ *be a* D*-stable prime ideal of* B *with* $\mathrm{ht}\,(\mathfrak{p}) = \dim X - 1$. *Then the closed set* $V(\mathfrak{p})$ *defined by* $\mathfrak{p}$ *is isomorphic to* $\mathbb{A}^1$.

(2) *Let* $\mathfrak{p}_i$ $(i = 1, 2)$ *be* D*-stable prime ideals of* B *with* $\mathrm{ht}\,(\mathfrak{p}_1) = \dim X - 1$. *If* $\mathfrak{p}_1 \neq \mathfrak{p}_2$ *then* $V(\mathfrak{p}_1) \cap V(\mathfrak{p}_2) = \emptyset$.

Proof. (1) Let $R = B/\mathfrak{p}$. By the hypothesis, $\dim R = 1$. Since $\mathfrak{p}$ is D-stable, D induces a surjective $\mathbb{C}$-derivation $\overline{D}$ on R. By Proposition 1.7 and Theorem 1.2, it follows that $V(\mathfrak{p}) \cong \mathbb{A}^1$.

(2) Suppose that $V(\mathfrak{p}_1) \cap V(\mathfrak{p}_2) \neq \emptyset$. Let $P \in V(\mathfrak{p}_1) \cap V(\mathfrak{p}_2)$. Then P corresponds to a maximal ideal $\mathfrak{m}$ of B such that $\mathfrak{m} \supset \mathfrak{p}_1 + \mathfrak{p}_2$. Since $\mathrm{ht}\,(\mathfrak{p}_1) = \dim X - 1$, we know that $\mathfrak{m}$ is a prime divisor of $\mathfrak{p}_1 + \mathfrak{p}_2$. Since $\mathfrak{p}_1 + \mathfrak{p}_2$ is D-stable, $\mathfrak{m}$ is also D-stable (cf. the proof of Lemma 2.2). Then D induces a non-trivial $\mathbb{C}$-derivation on $B/\mathfrak{m}$ because $D(t) = 1$. This is a contradiction because D is trivial on $\mathbb{C}$. □

Corollary 3.2. *With the notations and assumptions in Lemma 3.1, assume further that* $A := \mathrm{Ker}\, D$ *is finitely generated over* $\mathbb{C}$ *and* $\dim A = \dim B - 1$. *Let* $X = \mathrm{Spec}\, A$ *and let* $p : Y \to X$ *be the morphism associated to the inclusion* $A \hookrightarrow B$. *Then* p *is an* $\mathbb{A}^1$*-fibration. Furthermore, every one-dimensional fiber component of* p *is the affine line and disjoint from other fiber components.*

Proof. Since $Q(A)$ is algebraically closed in $Q(B)$ (cf. the proof of Lemma 2.1), p is an $\mathbb{A}^1$-fibration. In fact, for a maximal ideal $\mathfrak{m}$ of A, the ideal $\mathfrak{m}B$ is D-stable, and hence every minimal prime divisor of $\mathfrak{m}B$ is D-stable. Consider the fiber F of p over the closed point of X corresponding to $\mathfrak{m}$. If one fiber component F_1 of F has dimension one, then it is the affine line by Lemma 3.1, (1) and disjoint from the other fiber components of F by Lemma 3.1,(2). In particular, any general closed fiber is the affine line since $\mathfrak{m}B$ is then a prime ideal. By restricting X to an open set $D(f)$ of X with $f \in A$ and replacing X by $\mathrm{Spec}\, B[f^{-1}]$, we may assume that p is faithfully flat, X is smooth and every fiber of p is irreducible and reduced. Then p is an $\mathbb{A}^1$-bundle by [10, Theorem 2]. □

In a trial of reproving Cerveau's "theorem" in the case $\mathrm{Ker}\, D = \mathbb{C}$, we can prove the following result.

Theorem 3.3. *Let* R *be a normal affine domain of dimension one over* $\mathbb{C}$ *with a* $\mathbb{C}$*-derivation* D. *Let* t *be an indeterminate and let* $B = R[t]$. *Extend the derivation* D *to a* $\mathbb{C}$*-derivation* $\widetilde{D}$ *on* B *by setting* $\widetilde{D}(t) = 1$. *Assume that the following conditions are satisfied:*

(1) $\widetilde{D}$ *is surjective.*

(2) Ker $\widetilde{D} = \mathbb{C}$.

(3) *There exists a maximal ideal* $\mathfrak{m}$ *of* R *such that* $D(\mathfrak{m}) \subseteq \mathfrak{m}$.

Then R *is a polynomial ring* $\mathbb{C}[x]$ *and* $D = cx(d/dx)$ *after a suitable choice of* x *and with* $c \in \mathbb{C}^*$. *Hence* $B = \mathbb{C}[x, t]$ *and* $\widetilde{D} = cx(\partial/\partial x) + (\partial/\partial t)$.

Proof. We first prove the following

Claim 1. *Let* $\varphi : R \to \mathbb{C}$ *be the natural residue homomorphism of* R *to* $\mathbb{C} = R/\mathfrak{m}$. *Then* D *induces a bijective* $\mathbb{C}$*-linear mapping* $\mathfrak{m} \xrightarrow{\sim} \mathfrak{m}$.

For an element $a \in R$, there exists an element $h(t) \in B$ such that $\widetilde{D}(h(t)) = a$. Write

$$h(t) = a_0 + a_1 t + \cdots + a_n t^n,\ a_i \in R.$$

Then we have

$$a = D(a_0) + a_1,\ D(a_{i-1}) + i a_i = 0\ (1 < i \leq n),\ D(a_n) = 0.$$

Hence $a_n \in$ Ker D. By the condition (2), write $a_n = -c_n \in \mathbb{C}$. Then $D(a_{n-1}) + n a_n = 0$ implies that $\widetilde{D}(a_{n-1} - n c_n t) = 0$, whence $a_{n-1} = n c_n t + c_{n-1}$. Since t is an indeterminate over R, it follows that $c_n = 0$. Similarly, $D(a_{n-2}) + (n-1)a_{n-1} = 0$ implies that $c_{n-1} = 0$. This argument applies until the relation $D(a_1) + 2a_2 = 0$, and it follows that $a_2 = \cdots = a_n = 0$ and $a_1 \in \mathbb{C}$. Hence we have $a = D(a_0) + a_1$. The element a_1 is uniquely determined by a and $D(a_0) \in \mathfrak{m}$. In fact, if we have another expression $a = D(a'_0) + a'_1$ with $a'_0 \in R$ and $a'_1 \in \mathbb{C}$, then $D(a_0 - a'_0) = a'_1 - a_1$. Hence $a_0 - a'_0 = (a'_1 - a_1)t + c$ with $c \in \mathbb{C}$. This implies $a'_1 = a_1$. Write $\rho(a) = a_1$. On the other hand, write $a_0 = a' + c$ with $a' \in \mathfrak{m}$ and $c \in \mathbb{C}$. Then $D(a_0) = D(a') \in \mathfrak{m}$ by the condition (3). Hence the decomposition $a = D(a_0) + a_1$ is given by the $\mathbb{C}$-module decomposition $R = \mathfrak{m} \oplus \mathbb{C}$. Define a mapping $\rho : R \to \mathbb{C}$ by $\rho(a) = a_1$. Since $\mathfrak{m}$ is an ideal, ρ is a ring homomorphism and D induces a $\mathbb{C}$-linear mapping $D : \mathfrak{m} \to \mathfrak{m}$. It is injective since Ker $D \cap \mathfrak{m} = (0)$ by the condition (2). Furthermore, if $a \in \mathfrak{m}$, the above decomposition $a = D(a_0) + a_1$ gives $a_1 = 0$. Namely, $a = D(a_0)$. Thus D is surjective on $\mathfrak{m}$. We then prove the next

Claim 2. *Let* $\delta : \Omega_{R/\mathbb{C}} \to R$ *be the* R*-module homomorphism corresponding to the derivation* D. *Then* δ *induces an isomorphism between* $\Omega_{R/\mathbb{C}}$ *and* $\mathfrak{m}$. *Furthermore, every* 1*-form* $\omega \in \Omega_{R/\mathbb{C}}$ *is exact.*

In fact, since $d(a) = d(a + c)$ for every $c \in \mathbb{C}$, it follows that $\Omega_{R/\mathbb{C}}$ is generated by $\{da \mid a \in \mathfrak{m}\}$ as the R-module. Since $\delta(da) = D(a)$, it follows that Im δ is the R-ideal generated by $\{D(a) \mid a \in \mathfrak{m}\}$. By Claim

1, we know that $\operatorname{Im}\delta = \mathfrak{m}$. Since δ is clearly injective, δ induces an R-isomorphism between $\Omega_{R/\mathbb{C}}$ and $\mathfrak{m}$. Let $\omega \in \Omega_{R/\mathbb{C}}$. Then $\delta(\omega) = D(a)$ with $a \in \mathfrak{m}$. Hence $\delta(\omega - da) = 0$, and $\omega = da$.

Now, by the proof of Theorem 1.2, R is a polynomial ring $\mathbb{C}[x]$. Write $\mathfrak{m} = (x)$ after a suitable change of the variable x. Then $D(x) = xf(x)$ with $f(x) \in \mathbb{C}[x]$. Since $D : \mathfrak{m} \to \mathfrak{m}$ is bijective, there exists $h(x) \in \mathfrak{m}$ such that $D(h(x)) = x$. Then $h'(x)f(x) = 1$. Hence $f(x) = c \in \mathbb{C}^*$ and $D = cx(d/dx)$. Then it is clear that $\widetilde{D} = cx(\partial/\partial x) + (\partial/\partial t)$. □

If we drop the condition (3) in the assumptions of Theorem 3.3, R is no longer a polynomial ring as shown by the following example.

Example 3.4. *Let R be a Laurent polynomial ring $\mathbb{C}[x, x^{-1}]$ and let $D = x(d/dx)$. Let $B = R[t]$ be a polynomial ring in a variable t over R. Extend D to a $\mathbb{C}$-derivation $\widetilde{D} = x(\partial/\partial x) + (\partial/\partial t)$. Then $\widetilde{D}$ is surjective and* $\operatorname{Ker}\widetilde{D} = \mathbb{C}$, *but there is no maximal ideal $\mathfrak{m}$ of R such that $D(\mathfrak{m}) \subseteq \mathfrak{m}$.*

If the condition (3) above is not satisfied, we show that we are essentially in the situation of Example 3.4.

Theorem 3.5. *Let R be a normal affine domain of dimension one over $\mathbb{C}$ with a $\mathbb{C}$-derivation D. Let t be an indeterminate and let $B = R[t]$. Extend the derivation D to a $\mathbb{C}$-derivation $\widetilde{D}$ on B by setting $\widetilde{D}(t) = 1$. Assume that the following conditions are satisfied:*

(1) *$\widetilde{D}$ is surjective.*
(2) $\operatorname{Ker}\widetilde{D} = \mathbb{C}$.
(3) *There are no maximal ideals $\mathfrak{m}$ of R such that $D(\mathfrak{m}) \subseteq \mathfrak{m}$.*

Then R is a Laurant polynomial ring $\mathbb{C}[x, x^{-1}]$ and $D = cx(d/dx)$ after a suitable choice of x and with $c \in \mathbb{C}^*$. Hence $B = \mathbb{C}[x, x^{-1}, t]$ and $\widetilde{D} = cx(\partial/\partial x) + (\partial/\partial t)$.

Proof. Since $\widetilde{D}$ is surjective and $\operatorname{Ker}\widetilde{D} = \mathbb{C}$, the proof of Claim 1 above shows that $R = D(R) + \mathbb{C}$, where $D(R)$ is the image $\operatorname{Im} D$. Since any maximal ideal of R is not D-stable by the hypothesis, the ideal of R generated by $\{D(a) \mid a \in R\}$ is the unit ideal. Hence the R-homomorphism $\delta : \Omega_{R/\mathbb{C}} \to R$ corresponding to the derivation D is surjective. Since $\Omega_{R/\mathbb{C}}$ is a locally free R-module of rank 1, it follows that δ is an isomorphism. Since the $\mathbb{C}$-vector space $D(R)$ has codimension one in R, the $\mathbb{C}$-vector space $d(R) = \{da \mid a \in R\}$ in $\Omega_{R/\mathbb{C}}$ has also codimension one. By Grothendieck's algebraic de Rham theorem [8], $H^1(X; \mathbb{C}) = \mathbb{C}$, where $X = \operatorname{Spec} R$. Hence $b_1(X) = 1$. Then $X \cong \mathbb{A}^1_* = \operatorname{Spec}\mathbb{C}[x, x^{-1}]$, and the derivation D is of the form

$D = cx^m(d/dx)$, where $c \in \mathbb{C}^*$ and $m \in \mathbb{Z}$, for the derivation has no zeros. If $m \leq 0$, then $D((c^{-1}/(-m+1))x^{-m+1}) = \widetilde{D}(t) = 1$ and $(c^{-1}/(-m+1))x^{-m+1} - t \in \mathbb{C}$, which contradicts the hypothesis that t is an indeterminate over R. If $m \geq 2$, then $x \notin D(R)$, which is also a contradiction to $R = D(R) + \mathbb{C}$. Hence $m = 1$, and $\widetilde{D} = cx(\partial/\partial x) + (\partial/\partial t)$. □

Let B be an affine normal domain of dimension two over $\mathbb{C}$ and let D be a $\mathbb{C}$-derivation on B. An *integral curve* with respect to D is an irreducible curve C of $\operatorname{Spec} B$ defined by a prime ideal $\mathfrak{p}$ of height one such that $D(\mathfrak{p}) \subseteq \mathfrak{p}$. A nonzero element f of B is an *integral element* with respect to D if $D(f) = fh$ for some $h \in B$. We say that D has no *non-constant multiplicative characters* if $D(f) = fh$ with $f, h \in B \setminus \{0\}$ implies $h \in \mathbb{C}^*$.

Lemma 3.6. *Let B be the polynomial ring $\mathbb{C}[x, y]$ and let D be a surjective $\mathbb{C}$-derivation on B. Then the following assertions hold.*

(1) *Assume that an integral element f of B exists. Then f is written as a polynomial $f = F(x_2) \in \mathbb{C}[x_2]$, where x_2 is a coordinate of B and x_2 is an integral element.*

(2) *Assume that* $\operatorname{Ker} D = \mathbb{C}$ *and that D has no non-constant multiplicative characters. Then there exists at most one integral curve on* $\operatorname{Spec} B$.

(3) *With the same assumptions as in* (2) *above,* $\operatorname{Ker} D_K = \mathbb{C}$, *where K is the quotient field of B and D_K is the extension of D to K.*

Proof. (1) Let $f = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ be the irreducible decomposition of f. Then $p_1, \ldots, p_n$ are integral elements by Lemma 2.2. Since $\operatorname{Spec} B/(p_i) \cong \mathbb{A}^1$, it follows that p_i is a coordinate of B. We write $p_1 = x_2$ with $B = \mathbb{C}[x_1, x_2]$. For $i \neq 1$, $V(p_1) \cap V(p_i) = \emptyset$. Otherwise, there exists a D-stable maximal ideal of B, which is impossible. Write $p_i = x_2 g(x_1, x_2) + \varphi(x_1)$. Then $\varphi(x_1) = c \in \mathbb{C}^*$. Since p_i is also a coordinate, $p_i - c$ is irreducible and hence $g(x_1, x_2) \in \mathbb{C}^*$. So, f is a polynomial in x_2.

(2) Let $\mathfrak{p}_i = (u_i)$ define an integral curve for $i = 1, 2$. By (1) above, we may assume that $u_1 = x_2$ is a coordinate and $u_2 = x_2 + c$ with $c \in \mathbb{C}^*$. Since $D(x_2)$ is divisible by x_2, either $D(x_2) = 0$ or $D(x_2) = x_2 h$ with $h \in B \setminus \{0\}$. If $D(x_2) = 0$ then $x_2 \in \operatorname{Ker} D = \mathbb{C}$, which is a contradiction. In the latter case, since D has no non-constant multiplicative characters, it follows that $D(x_2) = \alpha x_2$ with $\alpha \in \mathbb{C}^*$. Replacing D by $\alpha^{-1} D$, we may assume that $D(x_2) = x_2$. Since $D(x_2) = D(x_2 + c) = \beta(x_2 + c)$ with $\beta \in \mathbb{C}^*$, we have $x_2 = \beta(x_2 + c)$, which is a contradiction.

(3) Let $\xi \in \operatorname{Ker} D_K$. Write $\xi = f/g$, where $f, g \in B$ and $\gcd(f, g) = 1$. If $g \in \mathbb{C}^*$, then $\xi \in \operatorname{Ker} D$ and hence $\xi \in \mathbb{C}^*$. Suppose that $g \notin \mathbb{C}^*$. Then $D(f) \in (f)$ and $D(g) \in (g)$. Every irreducible component of f, g is also D-stable. Hence there exist irreducible D-stable elements p, q of B such that $p \mid f$ and $q \mid g$. Since there is at most one integral curve on $\operatorname{Spec} B$, it follows from the foregoing assertion that $q = cp$ with $c \in \mathbb{C}^*$. This is a contradiction. □

Given a $\mathbb{C}$-derivation D on an affine domain B, the existence of an integral element is not clear. When B is a polynomial ring $\mathbb{C}[x, y]$, D being surjective implies the existence of an integral element. We show this result after Cerveau's argument [4], where an essential ingredient is a theorem of Dimca-Saito [5]. We denote by $\mathbb{C}[x_1, \ldots, x_n]$ the polynomial ring B in dimension n, by Ω^i the free B-module of differential i-forms $(0 \le i \le n)$ and by $\Omega^\bullet$ the differential complex $\{\Omega^i \ (0 \le i \le n); d : \Omega^i \to \Omega^{i+1}\}$, where d is defined by the exterior differentiation

$$d(f dx_{k_1} \wedge \cdots \wedge dx_{k_i}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{k_1} \wedge \cdots \wedge dx_{k_i}.$$

Given $f \in B$, define another differentiation D_f of $\Omega^\bullet$ by

$$D_f(\omega) = d\omega + df \wedge \omega, \quad \omega \in \Omega^i.$$

Then we have a crucial result of Dimca-Saito [5].

Lemma 3.7. *Let $f \in B$ and denote by $\varphi_f : \mathbb{A}^n \to \mathbb{A}^1$ the morphism induced by the inclusion $\mathbb{C}[f] \hookrightarrow B$. Let F be a general fiber of the morphism φ_f. Then there is an isomorphism*

$$H^{i+1}(\Omega^\bullet, D_f) \cong \widetilde{H}^i(F; \mathbb{C}) \quad \textit{for every } i,$$

where $\widetilde{H}^i(F; \mathbb{C})$ denotes the reduced cohomology.

Now we consider the case $n = 2$, which is due to Cerveau [4] and stated in the introduction as Theorem 2.

Theorem 3.8. *Let $D = f_1(\partial/\partial x_1) + f_2(\partial/\partial x_2)$ be a surjective derivation on $B = \mathbb{C}[x_1, x_2]$. Assume that* $\operatorname{Ker} \delta = \mathbb{C}$. *Then there exists an integral element with repsect to D.*

Proof. Set $\mathrm{vol} = dx_1 \wedge dx_2$ (the volume form) and set

$$\omega_D = i_D(\mathrm{vol}) = -f_2 dx_1 + f_1 dx_2.$$

Then we have

$$d\omega_D = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 \wedge dx_2.$$

Since D is surjective, there exists an element $g \in B$ such that

$$D(g) = f_1 \frac{\partial g}{\partial x_1} + f_2 \frac{\partial g}{\partial x_2} = -\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right).$$

Then we have

$$d\omega_D + dg \wedge \omega_D = 0, \quad \text{i.e.,}\, D_g(\omega_D) = 0.$$

Suppose that the curve $g = c$ is irreducible for a general $c \in \mathbb{C}$. Then a general fiber G of the morphism $\varphi_g : \mathbb{A}^2 \to \mathbb{A}^1$ is irreducible. Hence $\widetilde{H}^0(G; \mathbb{C}) = 0$. By Lemma 3.7, there exists an element $h \in B$ such that $D_g(h) = \omega_D$. Since $D_g(h) = d(h) + hdg$, we have

$$\omega_D = d(h) + hdg.$$

Note that the derivation D is given as $D = \delta \cdot d$ for a B-module homomorphism $\delta : \Omega^1 \to B$. In fact, $\delta(dx_1) = f_1$ and $\delta(dx_2) = f_2$. Hence we have $\delta(\omega_D) = -f_2 f_1 + f_1 f_2 = 0$. Thence we have

$$0 = \delta(\omega_D) = \delta(d(h) + hdg) = D(h) + hD(g).$$

Thus the element h is an integral element. Finally, the assumption that $g = c$ is irreducible for general $c \in \mathbb{C}$ (the generic irreducibility of g) is guaranteed by Cerveau [4, Section 6]. □

By Lemma 3.6, (1), the integral element $h \in B$ is an element of $\mathbb{C}[x_2]$, where x_2 is one of the coordinates. Since ω_D has no divisorial part, the polynomial $h(x_2)$ has only simple factors. Cerveau [4, Lemme 5.3] asserts that $h(x_2)$ is a linear polynomial. Thus we may assume that $\omega_D = c\omega'$ after a change of coodinates $x_2 \mapsto x_2 - \alpha$ with $\alpha \in \mathbb{C}$, where $\omega' = dx_2 + x_2 dg$.

Lemma 3.9. *With the assumptions and notations in the proof of Theorem 3.8, the following assertions hold.*

(1) *For every $\eta \in \Omega^1$, there exist $f \in B$ and $\eta_D \in \operatorname{Ker} \delta$ such that $\eta = \eta_D + df$* (see [4, Proposition 1.2]). *Furthermore, $\eta \wedge \omega_D = \delta(\eta) dx_1 \wedge dx_2$.*
(2) $\operatorname{Ker} \delta = B\omega_D$.
(3) *Let $\eta = h_1 dx_1 + h_2 dx_2$ be an element of* $\operatorname{Ker} \delta$. *If η has no divisorial part, then $\eta = c\omega_D$ with $c \in \mathbb{C}^*$.*
(4) *Let $\eta = f\omega_D$ be an element of* $\operatorname{Ker} \delta$. *Then we have*

$$d\eta + dg \wedge \eta = df \wedge \omega_D = D(f) dx_1 \wedge dx_2.$$

Hence $d\eta + dg \wedge \eta = 0$ if and only if $f \in \mathbb{C}$.

Proof. (1) The module Ker δ is a free B-submodule of rank one of Ω^1 which is a direct summand of Ω^1. This follows from the following exact sequence

$$0 \to \text{Ker}\,\delta \to \Omega^1 \xrightarrow{\delta} B \to 0.$$

Take $\eta = h_1 dx_1 + h_2 dx_2$ from Ω^1. Since D is surjective, there exists $f \in B$ such that $D(f) = h_1 f_1 + h_2 f_2$. Set $\eta_D = \eta - df$. Then $\eta_D \in \text{Ker}\delta$. The rest is easy to show.

(2) Let $\eta = h_1 dx_1 + h_2 dx_2$ be an element of Ker δ. Then $a\eta = b\omega_D$ with $a, b \in B$, $a \neq 0$ and $\gcd(a, b) = 1$. Then $ah_1 = -bf_2$ and $ah_2 = bf_1$. Since $\gcd(a, b) = 1$, we have $b \mid h_1$ and $b \mid h_2$. Writing $h_1 = bc$ and $h_2 = bd$ with $c, d \in B$, we have $f_1 = ad$ and $f_2 = -ac$. Since $\gcd(f_1, f_2) = 1$ as D is surjective, it follows that $a \in \mathbb{C}^*$. Hence $h_1 = -ba^{-1}f_2$ and $h_2 = ba^{-1}f_1$. Namely, $\eta = a^{-1}b\omega_D$.

(3) Suppose that η has no divisorial part, i.e., $\gcd(h_1, h_2) = 1$. Then $b \in \mathbb{C}^*$ as well. Hence $a^{-1}b \in \mathbb{C}^*$.

(4) Since we have $d\eta + dg \wedge \eta = df \wedge \omega_D + f(d\omega_D + dg \wedge \omega_D) = df \wedge \omega_D = D(f)dx_1 \wedge dx_2$, it is easy to obtain the conclusion. □

Replacing D by cD with $c \in \mathbb{C}^*$ if necessary, we may assume that $\omega_D = \omega' = dx_2 + x_2 dg$. It is shown in [4, Remarque 5.6, Lemmes 5.4 and 5.5] that g is of the form $W = \lambda x_1 + \varphi(x_2)$ with $\lambda \in \mathbb{C}^*$ and $\varphi(x_2) \in \mathbb{C}[x_2]$. But there is an elementary (computational) mistake in the proof of Lemma 5.5. Thus the assertion of Cerveau is yet to be proved.

Nowithstanding, we can show the following result.

Lemma 3.10. *With the above notations, especially with the notations in the proof of Theorem 3.8, the fibration* $\varphi_g : \mathbb{A}^2 \to \mathbb{A}^1$ *is an* $\mathbb{A}^1$*-fibration. Hence* g *is a ring generator of* B*.*

Proof. We use the result of Dimca-Saito and prove that the B-module homomorphism $D_g : \Omega^1_{B/\mathbb{C}} \to \Omega^2_{B/\mathbb{C}}$ is surjective. Since $\Omega^2 = Bdx_1 \wedge dx_2$, it suffices to show that $\varphi dx_1 \wedge dx_2 \in \text{Im}\, D_g$ for every $\varphi \in B$. Since D is surjective, we can choose $f \in B$ so that $\varphi = D(f)$. Set $\eta = f\omega_D$. By Lemma 3.9, (4), we have $D_g(\eta) = d\eta + dg \wedge \eta = D(f)dx_1 \wedge dx_2 = \varphi dx_1 \wedge dx_2$. Hence $\Omega^2 = \text{Im}\, D_g$. Let G be a general fiber of φ_g. By the foregoing argument, G is a smooth curve and $H^1(G; \mathbb{C}) = 0$ by Lemma 3.7. Hence G is isomorphic to $\mathbb{A}^1$, and φ_g is an $\mathbb{A}^1$-fibration. □

We summarize various obstacles as questions.

Question 3.11. Let $W \in B$. Is the ideal of B generated by $x_2\frac{\partial W}{\partial x_1}$ and $1+x_2\frac{\partial W}{\partial x_2}$ a unit ideal ? If necessary, we assume the condition that $D(x_2) = -x_2 D(W)$.

If the answer is positive, then $dx_2 + x_2 dW = c\omega_D$ with $c \in \mathbb{C}^*$ and $W - g$ is a constant.

Question 3.12. Let D be a surjective $\mathbb{C}$-derivation on $\mathbb{C}[x_1, x_2]$. Does D then have no non-constant multiplicative characters ?

If the assertion of Cerveau is verified, it gives a positive answer to the second question.

To finish this article, we give several examples of $\mathbb{C}$-derivations on $B = \mathbb{C}[x, y]$ which have no integral elements. We change here again the notations of variables from x_1, x_2 to x, y. This is to accord with the notations in the original sources which we refer to. A. Nowicki and other people [1, 2, 16] gave such examples of a derivation $D = (\partial/\partial x) + f(x, y)(\partial/\partial y)$ on B. In fact, D has no integral curves if $f(x, y)$ is one of the following polynomials

1 $f(x, y) = xy + 1$,
2 $f(x, y) = (x^2 + x)y + x^2$,
3 $f(x, y) = y^s + cx$ with $s \geq 2$ and $c \in \mathbb{C}^*$.

The following example is due to L. Makar-Limanov [11] and it seems to be new in construction. So, we include the computation which Makar-Limanov kindly communicated to us.

Example 3.13. *Let* $D = (x^2+\lambda y)(\partial/\partial x)+(y^2+\mu x)(\partial/\partial y)$ *on* $\mathbb{C}[x, y]$, *where* $\lambda, \mu \in \mathbb{C}^*$ *with* $\lambda^2 \neq \mu^2$. *Then* D *has no integral elements.*

Proof. Assume that $D(f) = gf$. Then $\deg_{x,y}(g) < 2$, where $\deg_{x,y}$ here signifies the total degree. Hence we may assume that $g = \alpha x + \beta y + \gamma$. Write $f = x^a f_0 + r$, where $\deg_x(r) < a$ and $f_0 \in \mathbb{C}[y]$. Then

$$D(f) = (af_0 + \mu f_0')x^{a+1} + \cdots = \alpha f_0 x^{a+1} + \cdots$$

and hence $af_0 + \mu f_0' = \alpha f_0$. Hence $\alpha = a$ and $f_0' = 0$. Similarly, $\beta = b = \deg_y(f)$ and $f = c'y^b + r'$ with $c' \in \mathbb{C}$ and $\deg_y(r') < b$.

Consider now the leading form $\overline{f}$ of f relative to the total degree. Write $\overline{f} = x^i y^j h$, where $h(x, 0)h(0, y) \neq 0$. We have $\overline{D}(\overline{f}) = (ax+by)\overline{f}$, where $\overline{D}(x) = x^2$ and $\overline{D}(y) = y^2$. Since $\overline{D}(x^i y^j) = (ix + jy)x^i y^j$, we have

$$\overline{D}(h) = \{(a - i)x + (b - j)y\}h.$$

Since $h = h_0 x^\ell + \cdots + h_\ell y^\ell$ with $h_0 h_\ell \neq 0$, we have $a-i = \ell, b-j = \ell$ and $\overline{D}(h) = \ell(x + y)h$. It is easy to see that all coefficients of h are defined uniquely by h_0 and that $h_0(x-y)^\ell = h$. Hence $\overline{f} = h_0 x^i y^j (x-y)^\ell$ and $a = \deg_x(f) = i+\ell, b = \deg_y(f) = j+\ell$. Since we know that the leading monomials of f relative to $\deg_x$ and $\deg_y$ are respectively cx^a and $c'y^b$ with $c, c' \in \mathbb{C}$, we can conclude that

$$i = j = 0,\ a = b = \ell,\ \overline{f} = (x-y)^a,\ D(f) = \{a(x+y)+\gamma\}f\ .$$

Introduce new variables $u = x-y$ and $v = x+y$. Then

$$\begin{aligned} D(u) &= uv + \frac{\lambda-\mu}{2}v - \frac{\lambda+\mu}{2}u, \\ D(v) &= \frac{1}{2}\{u^2+v^2+(\lambda+\mu)v+(\mu-\lambda)u\}, \\ D(f) &= (av+\gamma)f,\ \overline{f} = u^a,\ \deg(f) = a\ . \end{aligned}$$

Write $f = v^d\phi + \rho$, where $\deg_v(\rho) < d$. Then

$$\frac{d}{2}\phi + \left(u + \frac{\lambda-\mu}{2}\right)\phi' = a\phi$$

as the coefficients with v^{d+1} in the left and right sides of

$$D(f) = (av+\gamma)f.$$

Therefore we have

$$\phi = c\left(u+\frac{\lambda-\mu}{2}\right)^{a-\frac{d}{2}} \quad \text{and} \quad f = cv^d\left(u+\frac{\lambda-\mu}{2}\right)^{a-\frac{d}{2}} + \rho\ .$$

Hence $a = \deg(f) \geq d + a - \frac{d}{2}$ and $d = 0$. It means that

$$f = f(u) \quad \text{and} \quad D(f) = f'\left(uv + \frac{\lambda-\mu}{2}v - \frac{\lambda+\mu}{2}u\right) = (av+\gamma)f\ .$$

So, we have

$$\begin{aligned} &\frac{2uv+(\lambda-\mu)v-(\lambda+\mu)u}{av+\gamma} \\ &\quad = \frac{2u+\lambda-\mu}{a} - \frac{(2u+\lambda-\mu)\gamma + a(\lambda+\mu)u}{a(av+\gamma)} \in \mathbb{C}(u) \end{aligned}$$

which is possible only if $\gamma(\lambda-\mu) = 2\gamma + a(\lambda+\mu) = 0$. Since we assume that $\lambda^2 \neq \mu^2$, we have $\gamma = a = 0$. Namely $f \in \mathbb{C}$. □

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