ALGEBRAIC DERIVATIONS ON AFFINE DOMAINS

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To Professor R.V. Gurjar on his sixtieth birthday

Abstract. We define an *algebraic* derivation on an affine domain B defined over an algebraically closed field k of characteristic 0, which is called a *locally finite* derivation in [1], for example, and has appeared in commutative and non-commutative contexts in other references. We, without being aware of this existing definition and related results, introduced the term of algebraic derivation by extracting a property analogous to algebraic actions of algebraic groups. The first section is devoted to the graded ring structure which the algebraic derivation D defines on B in a natural fashion. The graded ring structure is indexed by an abelian monoid which is a submonoid of the additive group of the ground field k. This structure is already observed in [4]. But our approach is more computational and straightforward. If the monoid indexing the graded ring structure of B is rather restricted (see Theorem 1.9 below), the derivation D is close to what is called an *Euler derivation* mixed with a locally nilpotent derivation. We observe this fact when B is a polynomial ring mostly in dimension two. In fact, the results in section two give various characterization of a polynomial ring $k[x, y]$ in terms of algebraic derivations. The third section gives a remark on singularities which can coexist with algebraic derivations. The results given in sections two and three are new.

Introduction

Let G be an algebraic group defined over an algebraically closed field k. If G acts on an affine algebraic variety $X = \operatorname{Spec} B$, we say that the action is *algebraic* if for every element b of B the k-vector subspace $\sum_{g\in G(k)} k\cdot {}^g b$ in the ring B generated by all translates ${}^g b$ with $g \in G(k)$ has finite dimension, where $G(k)$ is the set of closed points of G. Suppose that G is an affine algebraic group $\operatorname{Spec} R$ and the G-action

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is given by a k-morphism $\sigma : G \times X \to X$. Let $\sigma^* : B \to R \otimes B$ be the coaction. For $b \in B$, write

$$\sigma^*(b) = \sum_{i=1}^{n} f_i \otimes b_i, \quad f_i \in R,\ b_i \in B.$$

Then ${}^g b = \sum_{i=1}^n \pi_g(f_i) \cdot b_i$, where $\pi_g : R \to k$ is the k-algebra homomorphism corresponding to a point $g \in G(k)$. Then $\sum_{g \in G(k)} k \cdot {}^g b$ is a k-subspace of $\sum_{i=1}^n k \cdot b_i$, whence $\sum_{g \in G(k)} k \cdot {}^g b$ has finite dimension.

Suppose that the ground field k has characteristic zero and G is the additive group scheme G_a. Then a G_a-action on an affine algebraic variety $X = \text{Spec}\, B$ corresponds bijectively to a locally nilpotent k-derivation δ on B. In this case the k-subspace $\sum_{g \in G_a(k)} k \cdot {}^g b$ coincides with $\sum_{n \geq 0} k \cdot \delta^n(b)$, which has finite dimension because $\delta^n(b) = 0$ for $n \gg 0$. Note that if ξ is an element of the quotient field $Q(B)$, the k-vector space $\sum_{n \geq 0} k \cdot \delta^n(\xi)$ does not necessarily have finite dimension. An example is a locally nilpotent derivation $\delta = \partial/\partial x$ on a polynomial ring $B = k[x]$. Taking $1/x$ as an element ξ, it follows that $\sum_{g \in G_a(k)} k \cdot {}^g \xi = \sum_{n \geq 0} k \cdot x^{-n}$, which is not of finite dimension.

We can extend the notion of algebraicity to regular vector fields on an affine algebraic variety. Namely, let D be a k-derivation on an affine domain B. We say that D is an *algebraic derivation* of B if for every element $b \in B$, the k-vector subspace $\sum_{n \geq 0} k \cdot D^n(b)$ has finite dimension. A typical example of such an algebraic derivation which is not locally nilpotent is the Euler derivation $D = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ on a polynomial ring $B = k[x_1, \ldots, x_n]$ and a composite of a derivation of Euler type and a locally nilpotent derivation.

Our objective in this article is to show that an algebraic derivation on an affine k-domain is something very close to the derivation of the above type. Furthermore, we try to prove some structure theorems on the affine domain provided B has an algebraic derivation. We first show that an algebraic derivation D on an affine domain B gives a graded ring structure on B (see Theorem 1.5). Then by making use of the graded ring structure and various properties of the algebraic derivations, we characterize polynomial rings of dimension two (see Theorems 2.4, 2.5 and 2.6). In section three, it is shown that the Euler derivation $D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on a polynomial ring $k[x, y]$ induces an algebraic derivation on $k[x, y]^G$ for every finite subgroup of $\text{GL}(2, k)$.

For an integral domain B, the quotient field of B is denoted by $Q(B)$ and the multiplicative group consisting of invertible elements of B by B^*. For a k-derivation D on a k-domain B, we denote by $\widetilde{D}$ the extension of D to $Q(B)$. We denote by Ker D and Ker $\widetilde{D}$ the kernels of D and $\widetilde{D}$ respectively, which are the subring of B and the subfield of $Q(B)$.

1. Algebraic derivations

Let k be an algebraically closed field of characteristic 0 and let B be an affine domain over k. Examples of algebraic derivations on B are

(1) a locally nilpotent derivation,
(2) a derivation of Euler type on a polynomial ring $B = k[x_1, \ldots, x_n]$.

$$D = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_n x_n \frac{\partial}{\partial x_n}, \quad \lambda_1, \ldots, \lambda_n \in k$$

Our first purpose is to give a structure theorem of an affine domain B with a non-trivial algebraic derivation D. Given a derivation D, we consider a k-algebra homomorphism $\varphi_D : B \to B[[t]]$ defined by

$$\varphi_D(b) = \sum_{n\geq 0} \frac{1}{n!} D^n(b) t^n \quad \text{for } b \in B.$$

Note that the homomorphism φ_D can be defined for any linear map $D : B \to B$.

For $\lambda \in k$, let $\Delta_\lambda = D - \lambda \cdot 1_B$, where 1_B signifies the identity morphism of B. We often write $\Delta_\lambda = D - \lambda$.

Lemma 1.1. *For every $\lambda \in k$, we have $\varphi_D = e^{\lambda t} \varphi_{\Delta_\lambda}$.*

Proof. Set $\Delta = \Delta_\lambda$ for simplicity. Then $D = \Delta + \lambda \cdot 1_B$. For $b \in B$, we compute as follows:

$$\begin{aligned}
\varphi_D(b) &= \sum_{n\geq 0} \frac{1}{n!}\ (\Delta + \lambda)^n(b)\ t^n \\
&= b\big(1 + \lambda t + \frac{1}{2!}\lambda^2 t^2 + \cdots + \frac{1}{n!}\lambda^n t^n + \cdots\big) \\
&\quad + \Delta(b) t \big(1 + \lambda t + \cdots + \frac{1}{(n-1)!}\lambda^{n-1} t^{n-1} + \cdots\big) + \cdots \\
&\quad + \frac{1}{m!}\Delta^m(b) t^m \big(1 + \lambda t + \cdots + \frac{1}{(n-m)!}\lambda^{n-m} t^{n-m} + \cdots\big) + \cdots \\
&= e^{\lambda t}\big\{b + \Delta(b) t + \frac{1}{2!}\Delta^2(b) t^2 + \cdots + \frac{1}{m!}\Delta^m(b) t^m + \cdots\big\} \\
&= e^{\lambda t}\varphi_\Delta(b).
\end{aligned}$$

□

For $\lambda \in k$, set

$$B_\lambda = \{b \in B \mid (D - \lambda)^n(b) = 0 \text{ for some positive integer } n\}.$$

Then it is clear that $b \in B_\lambda$ if and only if $\varphi_{\Delta_\lambda}(b) \in B[t]$. In particular,

$$B_0 = \{b \in B \mid \varphi_D(b) \in B[t]\}.$$

By making an essential use of Lemma 1.1, we prove the following lemmas.

Lemma 1.2. *We have the following assertions.*

(1) *B_0 is a k-subalgebra of B and D is locally nilpotent on B_0.*
(2) *For any $\lambda \in k$, B_λ is a B_0-module.*

Proof. (1) The assertion follows from the fact that φ_D is a k-algebra homomorphism.

(2) Let $a \in B_0$ and $b \in B_\lambda$. Then

$$\varphi_D(ab) = \varphi_D(a)\varphi_D(b) = e^{\lambda t}\varphi_D(a)\varphi_{\Delta_\lambda}(b).$$

On the other hand, we have $\varphi_D(ab) = e^{\lambda t}\varphi_{\Delta_\lambda}(ab)$. Hence it follows that $\varphi_{\Delta_\lambda}(ab) = \varphi_D(a)\varphi_{\Delta_\lambda}(b) \in B[t]$, which implies that $\Delta_\lambda^n(ab) = 0$ for some $n > 0$. □

Lemma 1.3. *For $\lambda, \mu \in k$, $B_\lambda B_\mu \subseteq B_{\lambda+\mu}$.*

Proof. Let $b \in B_\lambda$ and $c \in B_\mu$. Then

$$\varphi_D(bc) = \varphi_D(b)\varphi_D(c) = e^{(\lambda+\mu)t}\varphi_{\Delta_\lambda}(b)\varphi_{\Delta_\mu}(c)$$

and

$$\varphi_D(bc) = e^{(\lambda+\mu)t}\varphi_{\Delta_{\lambda+\mu}}(bc).$$

Hence $\varphi_{\Delta_{\lambda+\mu}}(bc) = \varphi_{\Delta_\lambda}(b)\varphi_{\Delta_\mu}(c) \in B[t]$. This implies that

$$(D - \lambda - \mu)^n(bc) = 0$$

for a sufficiently large n, and hence $bc \in B_{\lambda+\mu}$. □

Lemma 1.4. *If $\lambda \neq \mu$, then $B_\lambda \cap B_\mu = 0$.*

Proof. Suppose that k is a subfield of the complex number field $\mathbb{C}$. Let $b \in B_\lambda \cap B_\mu$. Then $\varphi_D(b) = e^{\lambda t}\varphi_{\Delta_\lambda}(b) = e^{\mu t}\varphi_{\Delta_\mu}(b)$. Hence we have

$$e^{(\lambda-\mu)t} = \frac{\varphi_{\Delta_\mu}(b)}{\varphi_{\Delta_\lambda}(b)},$$

where $\varphi_{\Delta_\lambda}(b), \varphi_{\Delta_\mu}(b) \in B[t]$. Hence the function in t on the right hand side is a rational function on $\mathbb{C}$. Meanwhile, the function on the left hand side is an entire function which does not have zeros and poles. Hence it follows that $\varphi_{\Delta_\mu}(b)$ and $\varphi_{\Delta_\lambda}(b)$ are constants, namely,

$(D-\lambda)(b)=0$ and $(D-\mu)(b)=0$. So, $D(b)=\lambda b=\mu b$, which implies $b=0$.

We may assume that k is a subfield of $\mathbb{C}$. Since B is finitely generated over k, B is isomorphic to the residue ring $k[x_1,\ldots,x_n]/(f_1,\ldots,f_m)$. Let $\alpha_1,\ldots,\alpha_r$ be the coefficients appearing in polynomials $f_1,\ldots,f_m \in k[x_1,\ldots,x_n]$, and let k_0 be the field $\mathbb{Q}(\alpha_1,\ldots,\alpha_r)$. Let

$$C=k_0[x_1,\ldots,x_n]/(f_1,\ldots,f_m).$$

Then $B=C\otimes_{k_0}k$. Adjoining to k_0 the elements λ,μ and the coefficients of $D(\overline{x}_i)$ for $1\le i\le n$ when they are expressed in the forms of polynomials in $\overline{x}_1,\ldots,\overline{x}_n$, where $\overline{x}_i$ is the residue class of x_i in B, we may assume that $\varphi_{\Delta_\lambda}(b)$ and $\varphi_{\Delta_\mu}(b)$ are polynomials in t with coefficients in k_0. Since k_0 is finitely generated over $\mathbb{Q}$, we may embed k_0 into $\mathbb{C}$ and replace k by the algebraic closure of the embedded k_0 in $\mathbb{C}$. Thus we may assume that $k\subseteq\mathbb{C}$. □

For an element $b\in B_\mu$, we define the *μ-height* of b as the non-negative integer r such that $(D-\mu)^r(b)\neq 0$ and $(D-\mu)^{r+1}(b)=0$. Note that μ-height is defined only for elements of B_μ. The first structure theorem on an affine domain with an algebraic derivation is stated as follows.

Theorem 1.5. *Let D be an algebraic derivation on an affine domain B. Then $B=\bigoplus_{\lambda\in k}B_\lambda$, which is a graded ring over B_0. Furthermore, $D(B_\lambda)\subseteq B_\lambda$.*

Proof. Let $b\in B$. Then the vector space $V=\sum_{n\ge 0}kD^n(b)$ has finite dimension. We may choose an integer $n>0$ so that

$$\{b, D(b), D^2(b),\ldots,D^{n-1}(b)\}$$

is a k-basis of V and $D^n(b)$ is expressed by a linear combination of $b, D(b),\ldots,D^{n-1}(b)$. Then D is a k-linear endomorphism on V and V decomposes into a direct sum $V=\oplus_\lambda V_\lambda$ where $V_\lambda=\{v\in V \mid (D-\lambda)^n(v)=0, n\gg 0\}$. Since $V_\lambda\subseteq B_\lambda$, it follows that $b\in\bigoplus_\lambda B_\lambda$. The previous lemmas show that $B=\bigoplus_\lambda B_\lambda$ is a graded ring over B_0. As for the second assertion, let $b\in B_\lambda$. Let r be the λ-height of b. If $r=0$, then $D(b)=\lambda b\in B_\lambda$. Suppose that $r>0$. Since $(D-\lambda)^{r+1}(b)=0$, the element $(D-\lambda)(b)\in B_\lambda$ and the λ-height of $(D-\lambda)(b)$ is $r-1$. By induction, we may assume $(D-\lambda)(b)\in B_\lambda$. Then $D(b)=\lambda b+(D-\lambda)(b)\in B_\lambda$. So, $D(B_\lambda)\subseteq B_\lambda$. □

We look into the properties of the subring B_0.

Lemma 1.6. *Let D be an algebraic derivation on a normal affine domain B. Then B_0 is integrally closed.*

Proof. Suppose that $\xi \in Q(B_0)$ is integral over B_0. Then it follows that $\xi \in B$ since B is normal. The element $\xi \in B$ satisfies

$$\xi^n + \alpha_1 \xi^{n-1} + \cdots + \alpha_n = 0,$$

where each α_i is an element of B_0. Hence it follows that

$$\varphi_D(\xi)^n + \varphi_D(\alpha_1)\varphi_D(\xi)^{n-1} + \cdots + \varphi_D(\alpha_n) = 0.$$

So, $\varphi_D(\xi)$ is integral over $k[\varphi_D(\alpha_1), \ldots, \varphi_D(\alpha_n)] \subset B[t]$. Since $B[t]$ is integrally closed and $\varphi_D(\xi) \in Q(B[t])$, it follows that $\varphi_D(\xi) \in B[t]$. Hence $\xi \in B_0$. □

We observe the following two examples. One deals with the Euler derivation and the other does a composite of the Euler derivation and a locally nilpotent derivation.

Example 1.7. Let $B = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables and let $D = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ be the Euler derivation on B. Then D is algebraic and $B = \bigoplus_{d \geq 0} B_d$, where B_d is the k-vector space spanned by monomials $x_1^{m_1} \cdots x_n^{m_n}$ of total degree d. In particular, $B_0 = k$. Meanwhile, $\mathrm{Ker}\,\widetilde{D}$ is $k\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$ which has transcendence degree $n-1$ over k. □

Example 1.8. Let $B = k[x, y]$ and let $D = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. For non-negative integers m and n, we have

$$D(x^n y^m) = n x^n y^m + m x^n y^{m-1}. \tag{1}$$

Hence for any non-negative integer r, we have $D^r(x^n y^m) \in k \cdot x^n y^m + k \cdot x^n y^{m-1} + \cdots + k x^n$. Thus it follows that the derivation D on B is algebraic. By (1), we have $(D - n)(x^n y^m) = m x^n y^{m-1}$ and

$$(D - n)^{m+1}(x^n y^m) = 0.$$

Hence if $f(y) \in k[y]$ with $\deg f(y) = m$, then

$$(D - n)^{m+1}(x^n f(y)) = 0$$

and $x^n f(y) \in B_n$. Since any element $b \in B$ is written as $b = f_0(y) + x f_1(y) + \cdots + x^n f_n(y)$ where $f_i(y) \in k[y]$ for $1 \leq i \leq n$, it follows that $B = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} B_n$ and $B_n = x^n k[y]$. The ring B is graded by the monoid $\mathbb{Z}_{\geq 0}$ of non-negative integers. □

Let D be an algebraic derivation on a normal affine domain B over k. Let $\Lambda = \{\lambda \in k \mid B_\lambda \neq 0\}$, which is a monoid under the addition of $(k, +)$, and let M be the abelian subgroup of $(k, +)$ generated by Λ. We call Λ (resp. M) the *monoid* (resp. *abelian group*) associated to D.

Theorem 1.9. *Under the notation and assumption as above, suppose that M is a totally ordered abelian group with ordering $<$ in the sense that $\lambda \geq 0$ for every $\lambda \in \Lambda$ and if $\lambda < \mu$ then $\lambda + \nu < \mu + \nu$ for any $\nu \in M$.* [1] *Then the following assertions hold.*

(1) $Q(B_0) \cap B = B_0$.
(2) $Q(B_0)$ *is algebraically closed in* $Q(B)$. *If* $B_0 \subsetneqq B$, *then we have* $\text{trans.deg}_k Q(B_0) < \text{trans.deg}_k Q(B)$.
(3) B_0 *is factorially closed in* B. *Hence if* B *is factorial, then so is* B_0.
(4) *Suppose that* B *is a polynomial ring of* $\dim \leq 3$. *Then* B_0 *is either a polynomial ring or* k.

Proof. (1) Suppose that $b \in Q(B_0) \cap B$. Then b is written as $a_0 b = a_1$ with $a_0, a_1 \in B_0$. By Theorem 1.5, $b = b_{\lambda_1} + \cdots + b_{\lambda_n}$ with $b_{\lambda_i} \in B_{\lambda_i}$ for $1 \leq i \leq n$. Since $a_0 b = a_0 b_{\lambda_1} + \cdots + a_0 b_{\lambda_n} = a_1$, it follows that $\lambda_i = 0$ for some i and $b = b_{\lambda_i} \in B_0$.

(2) Suppose that $\xi \in Q(B)$ is algebraic over $Q(B_0)$. Then there exists $a_0 \in B_0$ such that $a_0\xi$ is integral over B_0. Since $B_0 \subset B$ and B is integrally closed by assumption, $a_0\xi \in B$. The element $a_0\xi$ satisfies

$$(a_0\xi)^n + c_1(a_0\xi)^{n-1} + \cdots + c_n = 0 \tag{2}$$

where $c_i \in B_0$ for $1 \leq i \leq n$. Write

$$a_0\xi = b_{\lambda_1} + \cdots + b_{\lambda_m}, \quad b_{\lambda_1} \neq 0,\ b_{\lambda_m} \neq 0, \quad \lambda_1 < \cdots < \lambda_m.$$

If $\lambda_m > 0$, then the term $b_{\lambda_m}^n$, which has the highest order, cannot be cancelled by the other terms in the equation (2). Hence $\lambda_m \leq 0$. Similarly, $\lambda_1 \geq 0$. So, $a_0\xi \in B_0$ and $\xi \in Q(B_0)$. Suppose that $B_0 \subsetneqq B$. Then there exists an element $b \in B_\lambda$ with $\lambda \neq 0$. Then b is algebraically independent over $Q(B_0)$. For otherwise, b is algebraic over $Q(B_0)$ and hence $b \in Q(B_0)$. Then $b \in Q(B_0) \cap B = B_0$, which contradicts the choice of b.

(3) Suppose that $a = b_1 b_2 \in B_0$ with non-zero $b_1, b_2 \in B$. Write $b_1 = b_{\lambda_1} + \cdots + b_{\lambda_r}$ and $b_2 = b_{\mu_1} + \cdots + b_{\mu_s}$ where $b_{\lambda_i} \in B_{\lambda_i}$ and $b_{\mu_j} \in B_{\mu_j}$ with $\lambda_1 < \cdots < \lambda_r$ and $\mu_1 < \cdots < \mu_s$. We may assume that $b_{\lambda_1} b_{\lambda_r} b_{\mu_1} b_{\mu_s} \neq 0$. Then the highest term of $b_1 b_2$ is $b_{\lambda_r} b_{\mu_s}$ and the lowest is $b_{\lambda_1} b_{\mu_1}$. Hence $b_{\lambda_r} b_{\mu_s} = b_{\lambda_1} b_{\mu_1} = a$ and so, $b_1 = b_{\lambda_1}$ and $b_2 = b_{\mu_1}$. Since $\lambda_1 \geq 0$, $\mu_1 \geq 0$ and $\lambda_1 + \mu_1 = 0$, it follows that $b_1, b_2 \in B_0$ and B_0 is factorially closed in B.

(4) We may assume that $B_0 \subsetneqq B$ and $B_0 \neq k$. Then $\text{trans.deg}_k Q(B_0) < \text{trans.deg}_k Q(B) \leq 3$. Since $\dim B \leq 3$, it follows by the assertion (1)

[1] The condition that if $\lambda < \mu$ then $\lambda + \nu < \mu + \nu$ for any $\nu \in M$ follows from that M is a finitely generated subgroup of the additive group k.

and a result of Zariski [9] that B_0 is an affine domain. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} B_0$. Then the inclusion $B_0 \hookrightarrow B$ induces a dominant morphism $p : X \to Y$. Since B is factorial and $B^* = k^*$, B_0 is factorial by the assertion (3) and $B_0^* = k^*$. If $\dim B_0 = 1$, then B_0 is a polynomial ring. Suppose that $\dim B_0 = 2$. Then p is equi-dimensional over $p(X)$. In fact, suppose that there is an irreducible fiber component of dimension 2 in p. Then there exists a prime element $b \in B$ such that $bB \cap B_0 = \mathfrak{m}$ is a maximal ideal. Then $bb_1 = a \in B_0$. Since B_0 is factorially closed in B, it follows that $b \in B_0$. Hence $bB \cap B_0 = bB_0$, which is a contradiction. Then it follows that Y is isomorphic to $\mathbb{A}^2$ or a Platonic fiber space $\mathbb{A}^2/G$ (see [5, Theorem 3]). If G is nontrivial, p splits to $X \to \mathbb{A}^2 \to \mathbb{A}^2/G = Y$. In fact, Y has a unique singular point, say P and the smooth locus Y° has the universal covering $\mathbb{A}^2 \setminus \{O\}$, where O is the point of origin. Since $p^{-1}(P)$ is either empty or of dimension one, $\mathbb{A}^3 \setminus p^{-1}(O)$ is simply connected. Hence the fiber product

$$Z = (\mathbb{A}^3 \setminus p^{-1}(P)) \times_{Y^\circ} (\mathbb{A}^2 \setminus \{O\})$$

splits into a disjoint union of copies of $\mathbb{A}^3 \setminus p^{-1}(P)$. Then the restriction of the second projection $p_2 : Z \to \mathbb{A}^2 \setminus \{O\}$ to a connected component of Z provides the above splitting of p. This contradicts the fact that $Q(B_0)$ is algebraically closed in $Q(B)$. □

Example 1.10. Let $B = k[x, y]$ and

$$D = x^{n-1}y^n(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) \quad (n \geq 1).$$

We have by computation

$$\begin{aligned} D(x) &= x^n y^n,\ D^2(x) = 0 \quad \text{and} \\ D^k(y) &= (-1)^k k! x^{k(n-1)} y^{kn+1} \quad (k \geq 1) \end{aligned}$$

Thus $\sum_{j=0}^{\infty} k \cdot D^j(y)$ is infinite-dimensional, and D is not algebraic.

Next, compute $D^\ell(x^i y^j)$. For $0 \leq i < j$, we have

$$D^\ell(x^i y^j) = (i-j)(i-j-1)\cdots(i-j-\ell+1)x^{i+\ell(n-1)}y^{j+\ell n}.$$

For $i = j$, we have $D(x^i y^i) = 0$. For $i > j$, it follows that $x^i y^j = x^{i-j}(xy)^j \in B_0$. Hence $B_0 = k[x, xy]$. Therefore, we have $Q(B_0) = Q(B)$ and

$$Q(B_0) \cap B = Q(B) \cap B = B \supsetneqq B_0.$$

If we write $z = xy$, then $B_0 = k[x, z]$ and $D|_{B_0} = z^n \frac{\partial}{\partial x}$. □

We can raise the following natural question.

Question. Let B be an affine domain over k with an algebraic derivation D. Is B_0 finitely generated over k?

When D is locally nilpotent, it is clear that $B_0 = B$ is finitely generated. So, we are interested in the case that D is not a locally nilpotent derivation. If the monoid Λ is $\mathbb{Z}_{\geq 0}$, then B_0 is an affine domain and every B_μ is a finitely generated B_0-module.

An element λ of the monoid Λ is said to be *primitive* if $\lambda = \mu + \nu$ with $\mu, \nu \in \Lambda$ then either $\mu = 0$ or $\nu = 0$.

Lemma 1.11. *Let λ be a primitive element of the associated monoid Λ and let ξ be an element of B_λ such that $(D - \lambda)(\xi) = 0$. Suppose that the associated abelian group M is totally ordered, B is a factorial domain with $B^* = k^*$ and* $\operatorname{Ker} D = k$. *Then ξ is irreducible in B.*

Proof. Suppose that ξ is reducible. Write $\xi = b_1 b_2$ with $b_1, b_2 \in B$ and $\gcd(b_1, b_2) = 1$. Since ξ is a homogeneous element, both b_1 and b_2 are homogeneous. Since λ is primitive in Λ, we may assume that $b_1 \in B_0$ and $b_2 \in B_\lambda$. Since $(D - \lambda)(b_1 b_2) = 0$, we have

$$b_1(D(b_2) - \lambda b_2) = -b_2 D(b_1).$$

Since $\gcd(b_1, b_2) = 1$, $D(b_1)$ is divisible by b_1. By Theorem 1.9, B_0 is a factorial affine domain over k. Since $D(B_0) \subseteq B_0$ and D is locally nilpotent on B_0, it follows that $b_1 \in \operatorname{Ker} D = k$ (see [2]). This is a contradiction. □

2. Algebraic derivations on $k[x, y]$

Let B be an affine domain with an algebraic derivation D. Then D extends uniquely to a derivation $\tilde{D}$ on $K = Q(B)$. In the present section, we intend to characterize a two-dimensional polynomial ring $k[x, y]$ and algebraic derivations on $k[x, y]$ in terms of the subring B_0 and the associated monoid Λ, though our results are partial. We begin with the following result.

Lemma 2.1. *Let R be an affine domain of dimension one defined over a field K_0 of characteristic zero which is not necessarily algebraically closed. Assume that K_0 is algebraically closed in $Q(R)$ and that R is a graded ring $R = \bigoplus_{n \geq 0} R_n$ with $R_0 = K_0$. Then the following assertions hold.*

(1) *R is a polynomial ring $K_0[\xi]$ such that $R_n = K_0 \xi^n$ for every $n \geq 0$.*

(2) *Suppose that the graded ring structure on R is given by an algebraic K_0-derivation D and the associated monoid Λ is $\mathbb{Z}_{\geq 0}\lambda$ for some $\lambda \in \Lambda$. Then for every $\mu \in \Lambda$, we have*

$$R_\mu = \{\eta \in R \mid (D-\mu)(\eta) = 0\}.$$

Proof. (1) Let ξ be a nonzero element of R_1 and let $A = K_0[\xi]$. Then ξ is algebraically independent over K_0 and hence A is a graded subalgebra of dimension one in R. Since $\dim R = 1$, the field extension $Q(R) \supset Q(A)$ is algebraic. Let $\eta \in R_n$. Then there exist elements $f_0(\xi), f_1(\xi), \ldots, f_r(\xi) \in A$ with $\gcd(f_0(\xi), \ldots, f_r(\xi)) = 1$ such that

$$f_0(\xi)\eta^r + f_1(\xi)\eta^{r-1} + \cdots + f_r(\xi) = 0 \tag{3}$$

yields a minimal equation of η over $Q(A)$. Write

$$f_0(\xi) = \xi^m + (\text{terms of lower degree in } \xi).$$

Consider the homogeneous part of degree $(m+rn)$ in the equation (3) and obtain

$$\xi^m\eta^r + c_1\xi^{m+n}\eta^{r-1} + \cdots + c_r\xi^{m+nr} = 0, \quad c_1, \ldots, c_r \in K_0.$$

Hence we have

$$\eta^r + c_1\xi^n\eta^{r-1} + \cdots + c_r\xi^{nr} = 0. \tag{4}$$

By the minimality of the equation (3) over $Q(A)$, the equation (3) coincides with the equation (4) up to K_0^*. Hence $m = 0$ and the equation (3) is written as

$$\eta^r + c_1\xi^n\eta^{r-1} + \cdots + c_r\xi^{nr} = 0. \tag{5}$$

The equation (5) yields an algebraic equation

$$\left(\frac{\eta}{\xi^n}\right)^r + c_1\left(\frac{\eta}{\xi^n}\right)^{r-1} + \cdots + c_r = 0.$$

Since $\frac{\eta}{\xi^n}$ is an element of $Q(R)$ and K_0 is algebraically closed in $Q(R)$, it follows that $\frac{\eta}{\xi^n}$ is an element of K_0. This implies that $\eta \in K_0\xi^n$, and hence $R = A$.

(2) Note that λ is a minimal element of Λ other than 0. Let ξ be a nonzero element of R_λ such that $D(\xi) = \lambda\xi$. Then R is a polynomial ring $K_0[\xi]$ by the assertion (1). Let $\mu \in \Lambda$ and write $\mu = n\lambda$. Then every element η of R_μ is of the form $c\xi^n$ with $c \in K_0$. Hence we compute

$$D(\eta) = D(c\xi^n) = cn\xi^{n-1}D(\xi) = cn\lambda\xi^n = (n\lambda)\eta = \mu\eta.$$

Thence follows the assertion (2). □

The following result determines an algebraic derivation on $B = k[x]$.

Proposition 2.2. *Let $B = k[x]$ be a polynomial ring of dimension one and let D be a nontrivial algebraic derivation on B. Then, after a suitable change of variable, either $D = c\frac{d}{dx}$ or $D = cx\frac{d}{dx}$, where $c \in k^*$.*

Proof. If $B = B_0$ then D is locally nilpotent, and $D = c\frac{d}{dx}$ with $c \in k^*$. Suppose that $B \neq B_0$. Let $\xi \in B\backslash B_0$ such that $D(\xi) = \lambda\xi$ with $\lambda \in k^*$, and write $\xi = f(x) \in k[x]$. Then we have $D(\xi) = f'(x)D(x) = \lambda f(x)$. Since $\deg f(x) > 0$, it follows that $D(x)$ is a linear polynomial in x. Write $D(x) = cx + d$ with $c \neq 0$. Then $D(cx + d) = c(cx + d)$. By replacing x with $cx + d$, we may assume that $D(x) = cx$ with $c \in k^*$. Then $B = \bigoplus_{n\geq 0} B_n$ with $B_n = kx^n$ is the homogenoeous decomposition of B with respect to D. Hence $D = cx\frac{d}{dx}$. □

In the case $\dim B = 2$, if we assume that the monoid Λ associated to an algebraic derivation D is isomorphic to $\mathbb{Z}_{\geq 0}$, we have the following result.

Theorem 2.3. *Let B be an affine domain of dimension two. Then B is isomorphic to $k[x, y]$ if and only if the following conditions are satisfied.*

(1) *B is a factorial domain with $B^* = k^*$.*
(2) *B has a nontrivial algebraic k-derivation D such that the abelian group M generated by the associated monoid Λ is a totally ordered abelian group, $\dim B_0 \geq 1$ and Λ is isomorphic to $\mathbb{Z}_{\geq 0}$ provided $\dim B_0 = 1$.*

Proof. Since the "only if" part is easy, we prove only the "if" part. If $\dim B_0 = 2$ then $B = B_0$. In fact, if $B \supsetneq B_0$, take any element $\xi \in B\backslash B_0$. Then ξ is algebraic over $Q(B_0)$. Since $Q(B_0)$ is algebraically closed in $Q(B)$ and $Q(B_0) \cap B = B_0$ by Theorem 1.9, it follows that $\xi \in B_0$, a contradiction. Hence $B = B_0$. Then D is locally nilpotent on B. By an algebraic characterization of $k[x, y]$, B is a polynomial ring $k[x, y]$ [7, Theorem 2.2.1]. Suppose that $\dim B_0 = 1$. Then B_0 is an affine domain of dimension one such that B_0 is factorial and $B_0^* = k^*$. Hence $B_0 = k[x]$. By the assumption, $B = \bigoplus_{n\geq 0} B_{n\lambda}$. Let $K_0 = Q(B_0)$ and $R = B \otimes_{B_0} K_0$. Then R is an affine domain of dimension one over K_0 which has the graded ring structure $R = \bigoplus_{n\geq 0} R_n$ with $R_0 = K_0$, where $R_n = B_{n\lambda} \otimes_{B_0} K_0$. By Lemma 2.1, R is a polynomial ring $K_0[\xi]$. Hence the affine surface $\operatorname{Spec} B$ has an $\mathbb{A}^1$-fibration [3], and B is a polynomial ring in two variables by the algebraic characterization of $k[x, y]$ [7]. □

Under some additional conditions, we can further determine an algebraic derivation.

Theorem 2.4. *Let B be an affine domain of dimension two with an algebraic derivation D. Then $B = k[x, y]$ with*

$$D = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$$

for $\lambda \in k^*$ if and only if the following conditions are satisfied.

(1) *B is factorial and $B^* = k^*$.*
(2) $\mathrm{Ker}\, \widetilde{D} = \mathrm{Ker}\, D = k$ *and $B_0 \neq k$.*
(3) *The monoid Λ associated to D is the set $\mathbb{Z}_{\geq 0}\lambda$ of non-negative multiples of $\lambda \in k^*$.*

Proof. "Only if" part. Suppose that $B = k[x, y]$ and $D = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$. Then we can compute for every $\ell \geq 1$

$$D^\ell(x^m y^n) = n^\ell \lambda^\ell x^m y^n + \sum_{0 \leq r < m} c_r x^r y^n, \quad c_r \in k.$$

Hence $x^m y^n \in B_0$ if and only if $n = 0$. This implies that $B_0 = k[x]$. Further, $B_\mu \neq 0$ if and only if $\mu \in \mathbb{Z}_{\geq 0}\lambda$ and $B_{n\lambda} = B_0 y^n$ for a positive integer n. In order to find elements in $\mathrm{Ker}\widetilde{D}$, write an element $\xi \in Q(B)$ as $\xi = \frac{f}{g}$ with $f, g \in k[x, y]$ and $\gcd(f, g) = 1$. Then $\widetilde{D}(\xi) = 0$ implies $f \mid D(f)$ and $g \mid D(g)$. Write $f = a_0 + a_1 y + \cdots + a_m y^m$ with the $a_i \in k[x]$. Then it is straightforward to show that $f \mid D(f)$ only if f is a monomial in y with coefficients in k. Similarly, g is a monomial in y. Then $\widetilde{D}(\xi) = 0$ if and only if $\xi \in k$. The rest of the assertion is easy to show.

"If" part. By Theorem 2.3, B is a polynomial ring $k[x, y]$. If $B = B_0$, then D is locally nilpotent and $D = f(y)\frac{\partial}{\partial x}$ for a suitable choice of coordinates. It then follows that $\mathrm{Ker}\, D \neq k$, which is a contradiction. Hence $B \neq B_0$ and $\dim B_0 = 1$ by Theorem 1.9. Since B_0 is a factorial affine domain of dimension one with $B_0^* = k^*$, B_0 is a polynomial ring and we may put $B_0 = k[x]$. We may also assume that $D(x) = 1$. For every $\mu \in \Lambda$, there exists $\xi_\mu \in B_\mu$ such that $D(\xi_\mu) = \mu\xi_\mu$. In fact, take a nonzero element $z \in B_\mu$ and let r be the μ-height of z. Let $\xi_\mu = (D - \mu)^r z$. Then $\xi_\mu \in B_\mu$ satisfies $D(\xi_\mu) = \mu\xi_\mu$. We claim that for every $\mu \in \Lambda$

$$B_\mu = B_0 \xi_\mu.$$

Take any $\eta \in B_\mu$. If $(D - \mu)\eta = 0$, then $\widetilde{D}\big(\frac{\eta}{\xi_\mu}\big) = 0$. Hence $\frac{\eta}{\xi_\mu} \in \mathrm{Ker}\widetilde{D} = k$ and $\eta = c\xi_\mu$ with $c \in k$. We can show the claim by induction on the μ-height r of η. Then $(D - \mu)^r \eta \neq 0$ and $(D - \mu)^{r+1}\eta = 0$. By the induction hypothesis, there exists a polynomial $f(x) \in k[x]$ such that $(D-\mu)\eta = f(x)\xi_\mu$. Let $F(x) \in k[x]$ be a polynomial such that $F'(x) = f(x)$. Then

$$(D-\mu)(F(x)\xi_\mu) = F'(x)\xi_\mu + F(x)D(\xi_\mu) - \mu F(x)\xi_\mu = f(x)\xi_\mu.$$

Hence it follows that $(D-\mu)(\eta - F(x)\xi_\mu) = 0$, and $\eta - F(x)\xi_\mu = c\xi_\mu$ for $c \in k$. Thus the claim is verified. We can also verify by induction that $\deg f(x)$ is the μ-height of η when we write $\eta = f(x)\xi_\mu$.

Since $\Lambda = \mathbb{Z}_{\geq 0}\lambda$ by the hypothesis, it follows that $B_\mu = B_0\xi_\lambda^n$ if $\mu = \lambda n$. Hence $B = k[x, \xi_\lambda]$ and D is written as $D = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$ with $y = \xi_\lambda$. □

The following result differs from Theorem 2.4 only in the condition (2).

Theorem 2.5. *Let B be an affine domain of dimension two with a nontrivial algebraic derivation D. Then $B = k[x, y]$ with $D = \lambda y \frac{\partial}{\partial y}$ and $\lambda \in k^*$ if and only if the following conditions are satisfied.*

1. *B is factorial and $B^* = k^*$.*
2. $\mathrm{Ker}\, \widetilde{D} = Q(\mathrm{Ker}\, D) \supsetneqq k$.
3. *The monoid Λ associated with D is $\mathbb{Z}_{\geq 0}\lambda$ with $\lambda \in k^*$.*

Proof. "Only if" part. Clearly, $\mathrm{Ker}\, D = k[x] = B_0$ and $B = \bigoplus_{n \geq 0} B_0 y^n$ with $\mathrm{Ker}\, \widetilde{D} = k(x)$ and $\Lambda = \mathbb{Z}_{\geq 0}\lambda$.

"If" part. By the conditions (2) and (3), we have the inclusions

$$k \subsetneqq \mathrm{Ker}\, D \subseteq B_0 \subsetneqq B.$$

Hence $\dim \mathrm{Ker}\, D = \dim B_0 = 1$. Then every element of B_0 is algebraic over $\mathrm{Ker} D$. Since D is locally nilpotent over B_0, it follows that $\mathrm{Ker} D = B_0$ ([2]). By Theorem 1.9, $B_0 = k[x]$. Note that λ is a minimal element of Λ which is nonzero and $\Lambda = \mathbb{Z}_{\geq 0}\lambda$. Let ξ be an element of B_λ such that $(D-\lambda)\xi = 0$. By dividing ξ by elements of $k[x]$, we may assume that ξ is not divisible by any element of $k[x] \setminus k$. Let $\eta \in B_\mu$ where $\mu = n\lambda$. Suppose that $(D-\lambda)\eta = 0$. Then $\widetilde{D}\left(\frac{\eta}{\xi^n}\right) = 0$. Hence $\frac{\eta}{\xi^n} \in Q(B_0)$ and thus we find $f(x), g(x) \in B_0$ such that $f(x)\eta = g(x)\xi^n$ and $\gcd(f(x), g(x)) = 1$. Since B is factorial and ξ is not divisible by any element of $k[x] \setminus k$, it follows that $f(x) \in k$ and $\eta \in B_0\xi^n$.

We shall show that $B_\mu = \{\eta \in B \mid (D-\mu)(\eta) = 0\}$. Let $K_0 = Q(B_0)$ and $R = B \otimes_{B_0} K_0$. By Theorem 1.9, $Q(B_0)$ is algebraically closed in $Q(R)$. Then R is an affine domain of dimension one over K_0 and D extends to a K_0-trivial algebraic derivation on R because $D(x) = 0$. We may assume that $B \subset R$. Let $\eta \in B_\mu$. Then $\eta \in R_\mu$ and $(D-\mu)(\eta) = 0$ by Lemma 2.1.

Now we have proved that $R_\mu = B_0\xi^n$. Hence $B = k[x,\xi]$ and $D(\xi) = \lambda\xi$. So, it suffices to take $y = \xi$. □

We consider a characterization of an algebraic derivation of Euler type on a polynomial ring $k[x_1,\ldots,x_r]$. Let D be an algebraic derivation on an affine domain B. Let $\lambda_1,\ldots,\lambda_r$ be elements of k. We say that $\lambda_1,\ldots,\lambda_r$ are *numerically independent* if $n_1\lambda_1+\cdots+n_r\lambda_r = 0$ with the $n_i \in \mathbb{Z}$ implies $n_1 = \cdots = n_r = 0$. The monoid Λ of D is said to be a *free* monoid of *rank* r if there exists numerically independent elements $\lambda_1,\ldots,\lambda_r$ of k such that $\Lambda = \sum_{i=1}^r \mathbb{Z}_{\geq 0}\lambda_i$. Then the abelian group M generated by Λ is a free abelian group $\bigoplus_{i=1}^r \mathbb{Z}\lambda_i$ of rank r. Define a lexicographic order in M by setting

$$m_1\lambda_1+\cdots+m_r\lambda_r < n_1\lambda_1+\cdots+n_r\lambda_r$$

if there exists $1 \leq \ell \leq r$ such that $m_i = n_i$ for $1 \leq i < \ell$ and $m_\ell < n_\ell$. Then M is a totally ordered abelian group with the lexicographic order.

Theorem 2.6. *Let B be an affine domain of dimension r with a nontrivial algebraic derivation D. Then $B = k[x_1,\ldots,x_r]$ and $D = \sum_{i=1}^r \lambda_i x_i \frac{\partial}{\partial x_i}$ with numerically independent elements $\lambda_1,\ldots,\lambda_r \in k$ if and only if the following conditions are satisfied.*

1. *B is factorial and $B^* = k^*$.*
2. Ker $D = k$.
3. *The monoid Λ is a free monoid of rank r.*

Proof. The "only if" part is clear with an algebraic derivation $D = \sum_{i=1}^r \lambda_i x_i \frac{\partial}{\partial x_i}$, where $\lambda_1,\ldots,\lambda_r$ are numerically independent. The monoid Λ is then generated by $\lambda_1,\ldots,\lambda_r$ over $\mathbb{Z}_{\geq 0}$ and $B_\lambda = kx_1^{m_1}\cdots x_r^{m_r}$ if $\lambda = m_1\lambda_1+\cdots+m_r\lambda_r$.

We shall prove the "if" part. By the condition (3), the monoid is written as $\Lambda = \sum_{i=1}^r \lambda_i\mathbb{Z}_{\geq 0}$ with numerically independent elements $\lambda_1,\ldots,\lambda_r$ of k. For $1 \leq i \leq r$, choose elements $\xi_i \in B_{\lambda_i}$ such that $(D-\lambda_i)(\xi_i) = 0$. Since λ_i is primitive in Λ and the condition (2) is assumed, $\xi_1,\ldots,\xi_r$ are irreducible elements in B by Lemma 1.11. We claim that $\xi_1,\ldots,\xi_r$ are algebraically independent over k. In fact, let

$$\sum_{i_1,\ldots,i_r} c_{i_1\cdots i_r}\xi_1^{i_1}\cdots\xi_r^{i_r} = 0, \quad c_{i_1\cdots i_r} \in k$$

be an algebraic relation of $\xi_1,\ldots,\xi_r$ over k. Note that

$$\xi_1^{i_1}\cdots\xi_r^{i_r} \in B_{i_1\lambda_1+\cdots+i_r\lambda_r}$$

and $i_1\lambda_1+\cdots+i_r\lambda_r \neq j_1\lambda_1+\cdots+j_r\lambda_r$ in Λ whenever $(i_1,\ldots,i_r) \neq (j_1,\ldots,j_r)$. Since $B = \bigoplus_{\lambda\in\Lambda} B_\lambda$, it follows that

$$c_{i_1\cdots i_r} = 0 \quad \text{for} \quad \forall\ (i_1,\ldots,i_r).$$

Let $A = k[\xi_1, \ldots, \xi_r]$. Then A is a graded subalgebra of B of dimension r. Hence $Q(B)$ is algebraic over $Q(A)$. Let η be a homogeneous element of B_μ. Then there exist elements $a_0(\xi), \ldots, a_s(\xi)$ of A such that

$$a_0(\xi)\eta^s + a_1(\xi)\eta^{s-1} + \cdots + a_s(\xi) = 0. \tag{6}$$

We may assume that this gives a minimal equation of η over $Q(A)$ and $\gcd(a_0, \ldots, a_s) = 1$. We can write

$$a_0(\xi) = \xi_1^{m_1} \cdots \xi_r^{m_r} + (\text{lower terms in } \xi \text{ with the lexicographic order}).$$

Write $\mu = \alpha_1\lambda_1 + \cdots + \alpha_r\lambda_r$ with $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_{\geq 0}$. As in the proof of Lemma 2.1, the homogeneous part of degree

$$m_1\lambda_1 + \cdots + m_r\lambda_r + \mu s = (m_1 + \alpha_1 s)\lambda_1 + \cdots + (m_r + \alpha_r s)\lambda_r$$

in the equation (6) yields

$$\xi^m\eta^s + c_1\xi^{m+\alpha}\eta^{s-1} + \cdots + c_s\xi^{m+s\alpha} = 0, \tag{7}$$

where $c_1, \ldots, c_s \in k$ and $\xi^{m+i\alpha} = \xi_1^{m_1+i\alpha_1} \cdots \xi_r^{m_r+i\alpha_r}$ for $0 \leq i \leq s$. Hence $m_1 = \cdots = m_r = 0$ and the equation (7) yields an equation

$$\left(\frac{\eta}{\xi^\alpha}\right)^s + c_1\left(\frac{\eta}{\xi^\alpha}\right)^{s-1} + \cdots + c_s = 0. \tag{8}$$

Since k is algebraically closed and the equation (8) is a minimal equation, we have $\eta = c\xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r}$ with $c \in k$. Hence $B = A = k[\xi_1, \cdots, \xi_r]$. Let $x_i = \xi_i$ for $1 \leq i \leq r$. Then D is written as

$$D = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_r x_r \frac{\partial}{\partial x_r}.$$

□

If $\lambda_1, \ldots, \lambda_r$ are not numerically independent, we are ignorant of any characterization theorem corresponding to Theorem 2.6.

3. Algebraic derivations and singularities

Let B be a normal affine domain of dimension two. If D is a nontrivial locally nilpotent k-derivation on B, then $\operatorname{Spec} B$ has at most cyclic quotient singular points [6]. In the case of algebraic derivations, worse singularities can coexist as shown in the following result.

Theorem 3.1. *Let G be a finite group acting on the k-vector space $kx_1 + \cdots + kx_r$ linearly. Let B be a polynomial ring $k[x_1, \ldots, x_r]$ and let $A = B^G$ the ring of G-invariants. Let $D = \sum_{i=1}^r x_i \frac{\partial}{\partial x_i}$ be the Euler derivation. Then D induces a nontrivial algebraic derivation D_A on A.*

Proof. Note that B is a graded ring with grading $\deg x_i = 1$ for $1 \leq i \leq r$. Since G acts linearly on $kx_1 + \cdots + kx_r$, A is a graded subalgebra. Namely, $A = \bigoplus_{n \geq 0} A_n$, where $A_n = A \cap B_n$ with B_n the set of homogeneous polynomials of degree n. It is then straightforward to see that $D(f) = nf$ for every $f \in A_n$. Hence D induces a k-derivation D_A on A. Since D is algebraic, so is D_A. □

When $r = 2$, the Euler derivation in Theorem 3.1 induces an algebraic derivation on $k[x, y]^G$, where G is a finite subgroup of $\mathrm{GL}(2, k)$. However, we do not know the type of singularity which can coexist with algebraic derivations treated in Theorems 2.4 and 2.5.

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