Stability condition の空間について

信州大学　アソシエイト研究員
椎名　貴久

２００７年３月３日

Stability condition (Bridgeland)

$\mathcal{T}$: triangulated category

$\mathrm{Stab}(\mathcal{T})$: stability condition の空間

$\mathrm{Stab}(\mathcal{T}) \ni \sigma = (Z, \mathcal{P})$: stability condition

$Z : K(\mathcal{T}) \longrightarrow \mathbb{C}$, 群の準同型 ($K(\mathcal{T})$: Grothendieck 群)

$\mathcal{P}(\phi) \subset \mathcal{T}$: full additive subcategory, $\forall \phi \in \mathbb{R}$

1. $E \neq 0 \in \mathcal{P}(\phi) \;\Rightarrow\; Z(E) = m(E)\exp(i\pi\phi),\ m(E) \in \mathbb{R}_{>0}$

2. $\forall \phi \in \mathbb{R}$, $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

3. $\phi_1 > \phi_2, A_j \in \mathcal{P}(\phi_j) \;\Rightarrow\; \mathrm{Hom}(A_1, A_2) = 0$

4. $\forall E \neq 0 \in \mathcal{T}$, there exists $\phi_1 > \phi_2 > \cdots > \phi_n$ and

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

s.t. $A_j \in \mathcal{P}(\phi_j)$

Z : central charge

$\mathcal{P}(\phi)$ の 0 でない object : semistable

$\mathcal{P}(\phi)$ の simple な object : stable

$\boxed{\text{距離の定義}}$

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{T}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\}$$
$$\in [0, \infty]$$

$$(\phi^-_\sigma(E) = \phi_n,\ \phi^+_\sigma(E) = \phi_1)$$

$\boxed{\text{triangulated category}}$ として,

- abelian category の derived category
  - sheaf の category
    - $*$ Klein 特異点, A_n, D_n, E_6, E_7, E_8
    - $*$ K^3 surface

A_n 型 Klein 特異点の場合

G : 位数 $n+1$ の巡回群　　$(G \subset SL(2,\mathbb{C}))$

$f : X \longrightarrow \mathbb{C}^2/G$, minimal resolution

$\mathrm{Coh}(X)$: X の coherent sheaves からなる abelian category

$D^b(\mathrm{Coh}(X))$: $\mathrm{Coh}(X)$ の bounded derived category

$$\boxed{\mathcal{D} = \{E \in D^b(\mathrm{Coh}(X)) \mid \mathbf{R}f_*(E) = 0\}}$$

$$\boxed{\text{Klein 特異点の最小解消}} \xrightarrow{\text{双対グラフ}} \boxed{\text{Dynkin グラフ}} \longrightarrow \boxed{\text{preprojective algebra}}$$

$$\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}-\cdots-\overset{n}{\circ} \quad A_n \qquad\qquad\qquad P[A_n]$$

Theorem

$P[A_n]$mod : abelian category of fin. gen. $P[A_n]$-modules

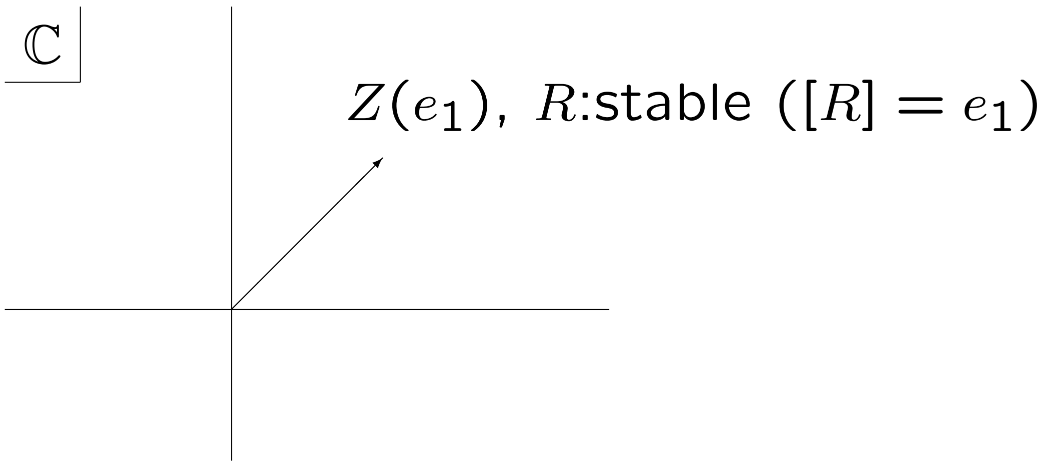
$$D^b(P[A_n]\text{mod}) \cong \mathcal{D} \text{ (as triang. category)}$$

Stab($\mathcal{D}$) について

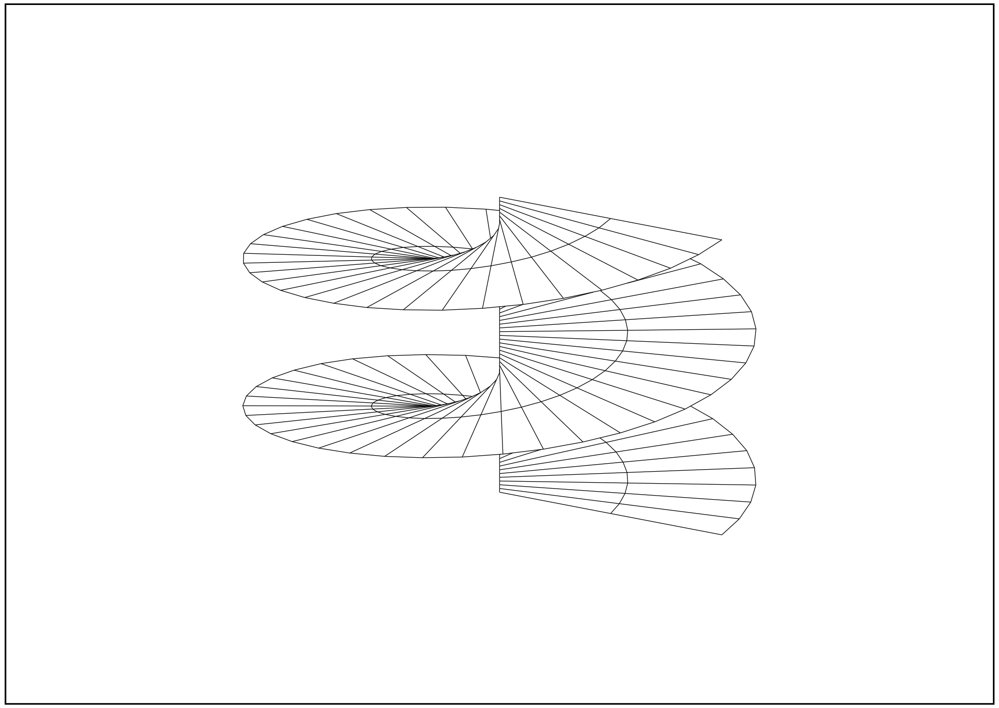
$S_1, \ldots, S_n$: $P[A_n]$-mod の simple な(互いに同型でない) objects

$K(\mathcal{D}) = \mathbb{Z}\langle e_1, \ldots, e_n \rangle$, 自由アーベル群

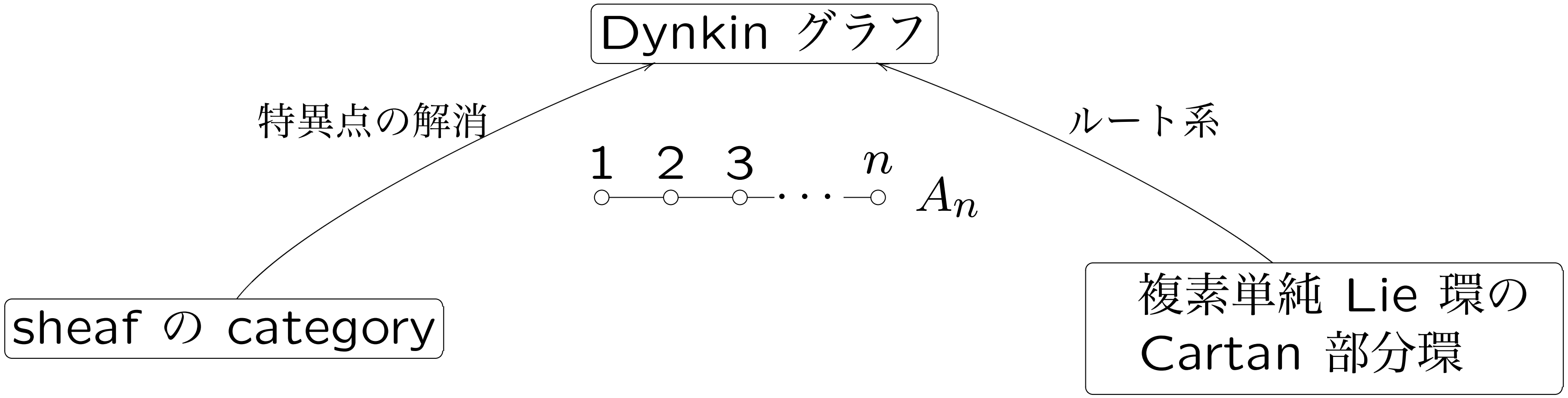
—— $n = 1$ のときの Stab($\mathcal{D}$) —— Stab($\mathcal{D}$) $\ni \sigma = (Z, \mathcal{P})$



$$
\begin{aligned}
\mathrm{Stab}(\mathcal{D}) &= \{(Z(e_1), \text{stable な object のとり方})\} \\
&= \{(r\mathrm{exp}(i\pi\phi), t) \in \mathbb{C} \times \mathbb{R} \mid r > 0, \phi = t\}
\end{aligned}
$$



Dynkin グラフとその周辺



Stab($\mathcal{D}$)

$\mathfrak{h}^{\mathrm{reg}}/W$

複素単純 Lie 環の Cartan 部分環 (A_n)

$\mathfrak{h} = \left\{(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum z_i = 0\right\}$, Cartan 部分環

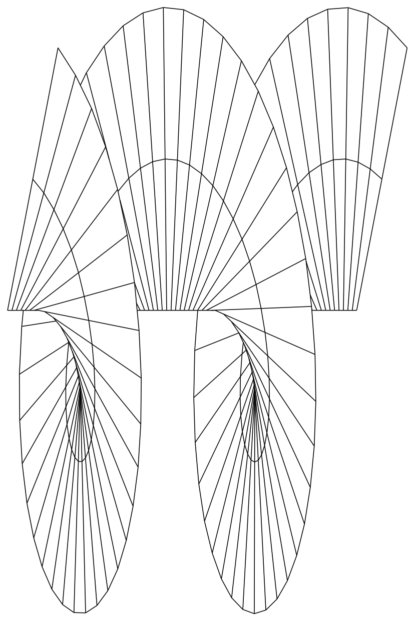
$\mathfrak{h}^{\mathrm{reg}} = \{(z_1, z_2, \ldots, z_{n+1}) \in \mathfrak{h} \mid z_i \neq z_j \text{ if } i \neq j\}$

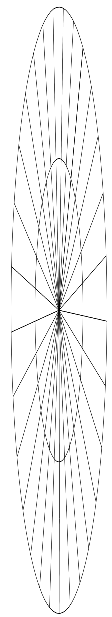
Σ_{n+1} ($n+1$ 次対称群) が $\mathfrak{h}^{\mathrm{reg}}$ に作用

Theorem. (Ishii-Ueda-Uehara)

$\mathrm{Stab}(\mathcal{D}) \longrightarrow \mathfrak{h}^{\mathrm{reg}}/\Sigma_{n+1}$, universal covering space

$\boxed{n = 1 \text{ のとき}}$

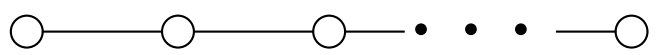




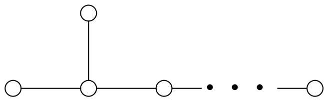
$\mathsf{Stab}(\mathcal{D})$

$\mathfrak{h}^{\mathsf{reg}}/\Sigma_2 = \mathbb{C}\backslash\{0\}$

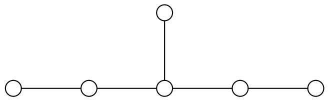
他の Klein 特異点と Dynkin グラフ



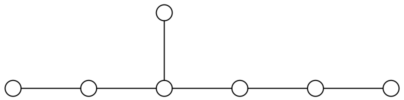
A_n



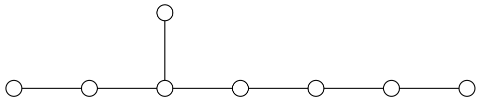
D_n



E_6



E_7



E_8

Remark.

D, E 型の stability condition の空間の一部分が covering space

universal？ 連結？

Stab を調べる方針 (目標)

Stability condition の空間の芯を作る.

Remark.

$\mathfrak{h}^{\mathrm{reg}}$ については, その領域の組み合わせから, 芯となるものができる.

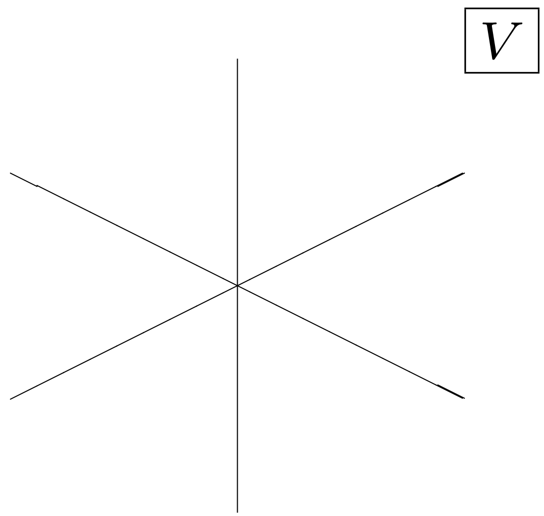
$\longrightarrow$ Salvetti complex

データ

V : 実 vector 空間

$\mathcal{A} = \{H \mid V$ の超平面$\}$, hyperplane arrangement

$$\mathcal{M}(\mathcal{A}) = V \otimes \mathbb{C} \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbb{C}$$



Salvetti complex

$$\begin{tikzcd}
\boxed{\text{arrangemet}} \arrow[r, "\text{超平面により}", "\text{部屋に別れる}"'] & \boxed{\text{chamber}} \arrow[d, "\text{chamber と壁の組み合わせ}"] & \\
& \boxed{\text{単体複体}} \arrow[r, equal] & \boxed{\mathsf{Sal}(\mathcal{A})}
\end{tikzcd}$$

Theorem.

$\mathsf{Sal}(\mathcal{A}) \hookrightarrow \mathcal{M}(\mathcal{A})$, 変位レトラクト

例： A_n の場合

$$\mathfrak{h}_{\mathbb{R}} = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \sum z_i = 0 \right\}$$

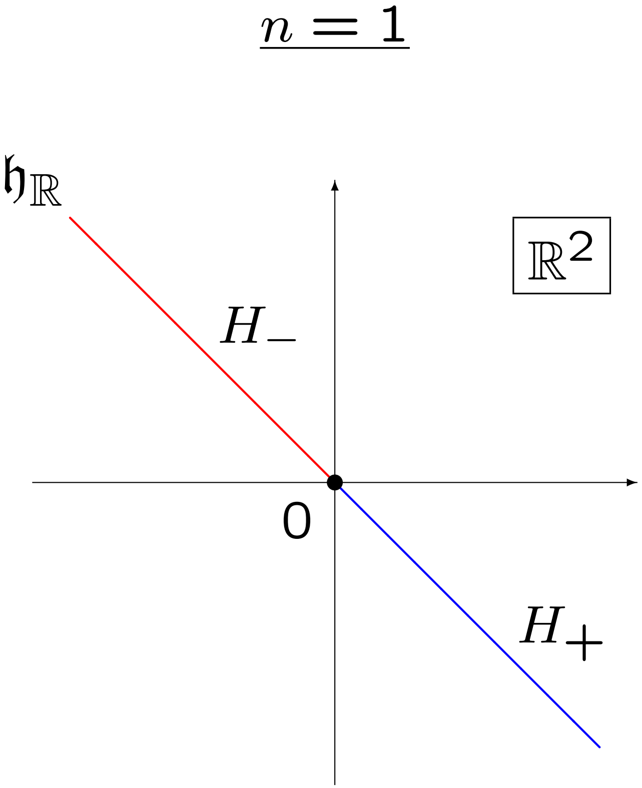
$$\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}} = \{(z_1, \ldots, z_n) \in \mathfrak{h}_{\mathbb{R}} \mid z_i \neq z_j \text{ if } i \neq j\}$$

$$H_{i,j} = \{(z_1, \ldots, z_{n+1}) \in \mathfrak{h}_{\mathbb{R}} \mid z_i = z_j\}$$

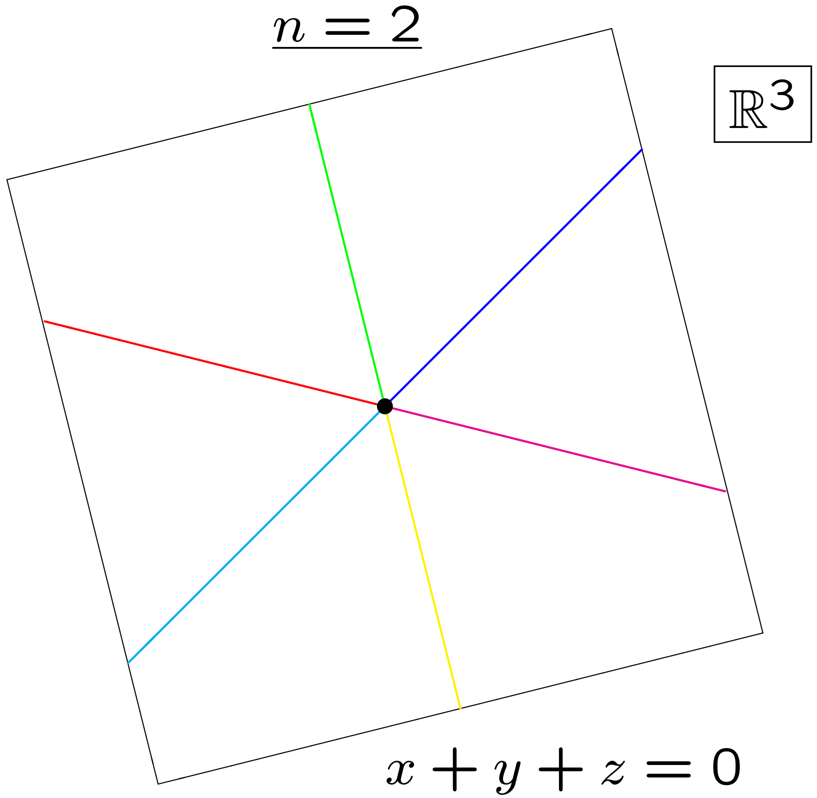
$$\boxed{\mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}} = \mathfrak{h}_{\mathbb{R}} \setminus \bigcup H_{i,j}}$$

$$V = \mathfrak{h}_{\mathbb{R}}^{\mathrm{reg}},\ \mathcal{A} = \{H_{i,j}\}$$

$\mathcal{M}(\mathcal{A}) = \mathfrak{h}^{\mathrm{reg}}$ ($\leftarrow$ 複素数のもの)



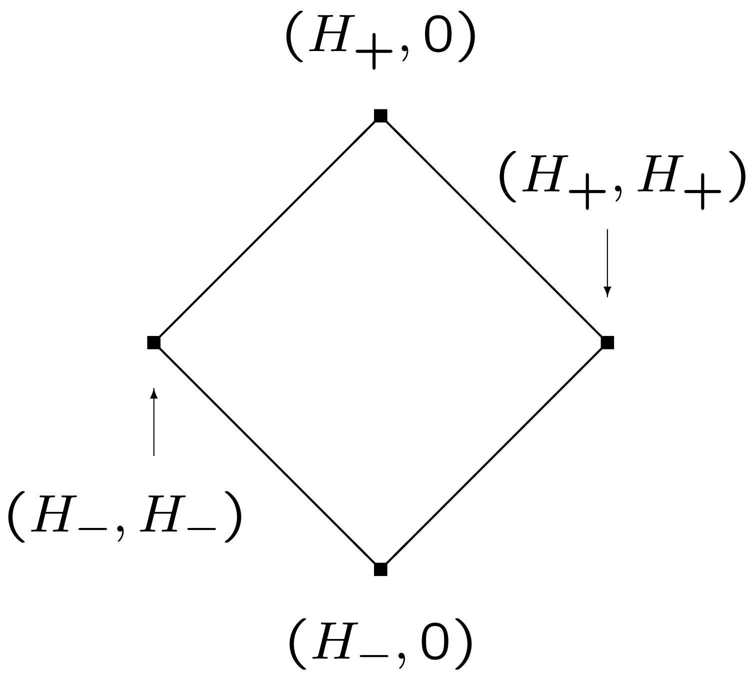
$\underline{n=1}$
$\mathfrak{h}_{\mathbb{R}}$
$\boxed{\mathbb{R}^2}$
H_-
0
H_+

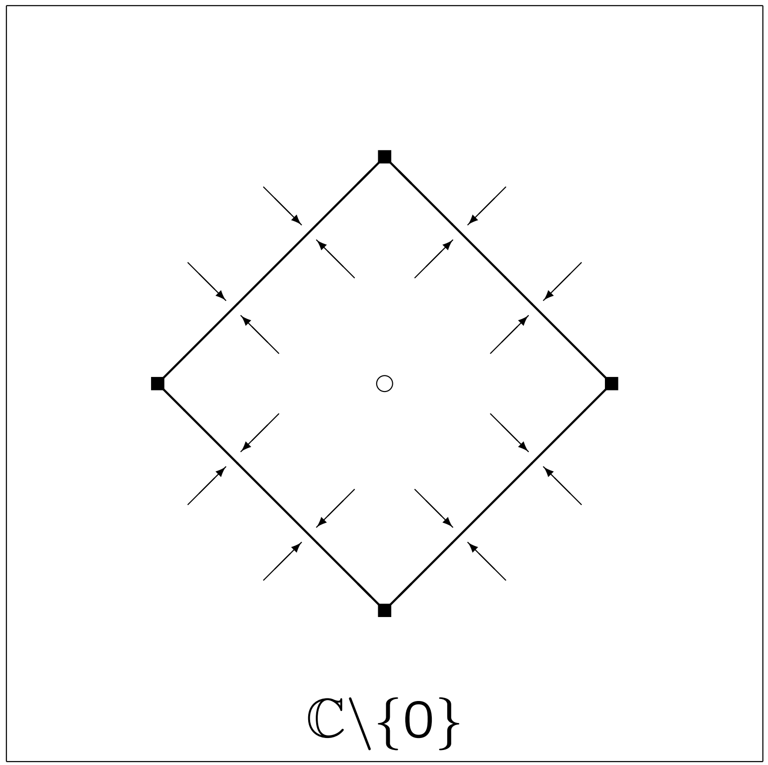


$\underline{n=2}$
$\boxed{\mathbb{R}^3}$
$x+y+z=0$

$\boxed{n = 1 \text{ のときの } \mathsf{Sal}(\mathcal{A})}$

$\mathsf{Sal}(\mathcal{A}) \to \mathcal{M}(\mathcal{A}) = \mathfrak{h}^{\mathsf{reg}} = \mathbb{C}\backslash\{0\}$

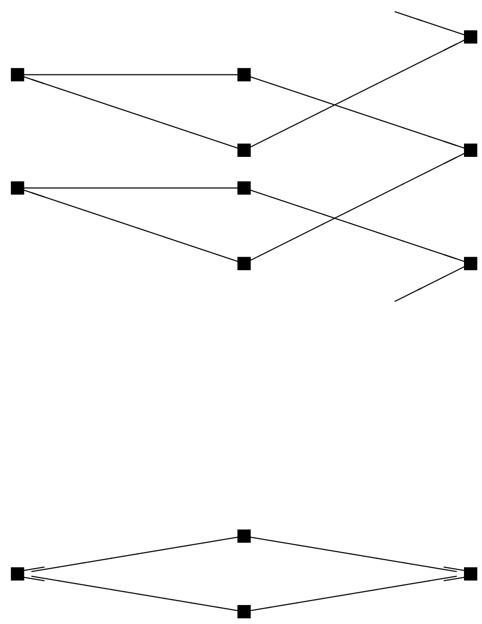




目標

Salbvetti conplex の universal cover を stability condition の空間に埋め込みたい(stability condition の芯)

$$\begin{tikzcd}
\mathrm{Stab}(\mathcal{D}) \arrow[d] & \widetilde{\mathrm{Sal}(\mathcal{A})} \arrow[l, dotted] \arrow[d] \\
\mathfrak{h}^{\mathrm{reg}} & \mathrm{Sal}(\mathcal{A}) \arrow[l]
\end{tikzcd}$$



$\boxed{\text{芯を見る事によって Stab のことを知る}}$ (universal, 連結)

問題

- どのように埋め込むか？(頂点としての stability condition の選び方)

- 他の Klein 特異点の場合は？(芯から分かる stability condition の性質？)

- もっと一般の triangulated category については？