# Holonomic D-modules and Noetherian differential operators for local cohomology classes associated to primary ideals

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Niigata, Japan

Nov. 2024, Kyoto

# introduction

Aim:

comlex analysis of hypersurfaces with non-isolated singularities

via holonomic D-modules

New

effective method for analizing of the structure of holonomic D-modules

associated to the Bernstein-Sato polynomial of

a hypersurface with non-isolated singularities

**Key**

The concept of Noetherian differential operators for

local cohomology classes associated to a primary ideal

# introduction

Main results of this talk

The concept of Noetherian differential operators

basic ideas

Grothendieck local duality with differential operators

+ extension by max. independent set

byproduct

A new alternative, effective algorithm for computing

classical Noether operators for primary ideals

in the sence of Ehrenpreis-Palamodov

# contents

# local duality and differential operators

$\Omega^n_X$: sheaf on $X$ of holomorphic differential $n$-forms

$X \subset \mathbb{C}^n$: open nbhd of the origin $O \in \mathbb{C}^n$

$\mathcal{H}^n_{\{O\}}(\Omega^n_X)$: local cohomology supported at $O \in X$

parfect pairing

$$\mathrm{res} : \mathcal{O}_{X,O} \times \mathcal{H}^n_{\{O\}}(\Omega^n_X) \longrightarrow \mathbb{C}$$

$\mathcal{H}^n_{[O]}(\Omega^n_X)$: algebric local cohomology supported at $O \in X$

$\mathcal{H}^n_{[O]}(\Omega^n_X) = \lim_{k\to\infty} \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{m}^k, \Omega^n_X)$

$\mathfrak{m}$: the maximal ideal at $O \in X$

parfect pairing

$$\mathrm{res} : \hat{\mathcal{O}}_{X,O} \times \mathcal{H}^n_{[O]}(\Omega^n_X) \longrightarrow \mathbb{C}$$

# local duality and differential operators

$F = [f_1, f_2, \ldots, f_k]$: a list, $k \geqslant n$, $X \subset \mathbb{C}^n$: a nbhd of O

$f_1, f_2, \ldots, f_k$: holomorphic functions on $X$

$V(F) \cap X = \{O\}$

$I_F \subset \mathcal{O}_{X,O}$, $\hat{I}_F \subset \hat{\mathcal{O}}_{X,O}$: ideals generated by F

$$W_F = \{\omega \in \mathcal{H}^n_{\{O\}}(\Omega^n_X) \mid I_F \omega = 0\}$$

$$\hat{W}_F = \{\omega \in \mathcal{H}^n_{[O]}(\Omega^n_X) \mid \hat{I}_F \omega = 0\}$$

non-degenerate pairing (Grothendieck local duality)

$$\text{res} : \mathcal{O}_{X,O}/I_F \times W_F \longrightarrow \mathbb{C}$$

$$\text{res} : \hat{\mathcal{O}}_{X,O}/\hat{I}_F \times \hat{W}_F \longrightarrow \mathbb{C}$$

Notice that $W_F \cong \hat{W}_F$ and thus $\mathcal{O}_{X,O}/I_F \cong \hat{\mathcal{O}}_{X,O}/\hat{I}_F$

# local duality and differential operators

Grothendieck local duality implies

$$I_F = \mathcal{A}nn_{\mathcal{O}_{X,O}}(W_F)$$

Let

$\{\omega_1, \omega_2, \ldots, \omega_m\}$ a basis, as a vector space over $\mathbb{C}$, of $W_F$

$h \in \mathcal{O}_{X,O}$: a germ of holomorphic function

then,

$h \in I_F$, iff

$$\mathrm{res}(h, \omega_j) = 0, \ j = 1, 2, \ldots, m$$

These conditions are computable !!

# local duality and differential operators

Let $H_{I_F} = \{\psi \in H^2_{[O]}(\mathcal{O}_X) \mid I_F\psi = 0\}$, then

$$W_{I_F} = \{\psi \cdot dx \wedge dy \mid \psi \in H_{I_F}\}$$

**Example** $f(x,y) = x^3 + y^7 + xy^5$, $F = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}]$

$\dim_{\mathbb{C}}(\mathcal{O}_{X,O}/I_F) = 12$

A basis of the vector space $H_{I_F}$;

$$\begin{bmatrix} 1 \\ xy \end{bmatrix}, \begin{bmatrix} 1 \\ xy^2 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^3 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y \end{bmatrix}, \begin{bmatrix} 1 \\ xy^4 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^2 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^5 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ x^2y^3 \end{bmatrix}, \begin{bmatrix} 1 \\ xy^6 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ x^3y \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^4 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^5 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ xy^7 \end{bmatrix}$$

$$-\frac{5}{21}\begin{bmatrix} 1 \\ x^3y^2 \end{bmatrix}, \begin{bmatrix} 1 \\ x^2y^6 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ x^4y \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ xy^8 \end{bmatrix} - \frac{5}{21}\begin{bmatrix} 1 \\ x^3y^3 \end{bmatrix}$$

## contents

# local duality and differential operators

Let $h(x,y) = \sum_{i,j} h_{i,j} x^i y^j \in \mathcal{O}_{X,O}$

for $w_2 = \begin{bmatrix} 1 \\ xy^2 \end{bmatrix} dx \wedge dy$ ,  $\mathrm{res}(h, w_2) = h_{0,1}$

for $w_9 = (\begin{bmatrix} 1 \\ xy^6 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ x^3y \end{bmatrix}) dx \wedge dy$, $\mathrm{res}(h, w_9) = h_{0,5} - \frac{1}{3}h_{2,0}$

Likewise $h(x,y) \in I_F$, iff

$$h_{0,0} = h_{0,1} = h_{0,2} = h_{1,0} = h_{0,3} = h_{1,1} = h_{0,3} = h_{1,2} = 0$$
$$h_{0,5} - \frac{1}{3}h_{2,0} = h_{1,3} = h_{1,4} - \frac{5}{7}h_{0,6} - \frac{5}{21}h_{2,1} = 0$$
$$h_{1,5} - \frac{5}{7}h_{3,1} - \frac{5}{7}h_{0,7} - \frac{5}{21}h_{2,2} = 0,$$ 12 linear conditions

Since $h_{i,j} = \frac{1}{(i-1)!(j-1)!}\frac{\partial^{i+j}h}{\partial x^i \partial y^j}(0,0)$, the condition can be rewritenn

in terms of differential operators !!

# local duality and differential operators

Let

$\mathcal{D}_X$: sheaf on $X$ of linear partial differential operators

$\mathcal{O}_X$: a left $\mathcal{D}_X$-module

$\mathcal{H}^n_{[O]}(\Omega^n_X)$: a right $\mathcal{D}_X$-module

For $P(x, \frac{\partial}{\partial x}) \in \mathcal{D}_X$, $\omega = \psi dx$, with $\psi \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$

$\omega P = (P^*\psi)dx$, $P^*$: the formal adjoint of $P$.

Example

for $w_1 = \begin{bmatrix} 1 \\ xy \end{bmatrix} dx \wedge dy$, $w_2 = \begin{bmatrix} 1 \\ xy^2 \end{bmatrix} dx \wedge dy$ , we have

$(-\frac{\partial}{\partial y}) \begin{bmatrix} 1 \\ xy \end{bmatrix} = \begin{bmatrix} 1 \\ xy^2 \end{bmatrix}$, namely $w_2 = w_1 \frac{\partial}{\partial y}$,

# local duality and differential operators

for $h(x,y) = \sum_{i,j} h_{i,j} x^i y^j \in \mathcal{O}_{X,O}$

$$h_{0,1} = \operatorname{res}(h, \begin{bmatrix} 1 \\ xy^2 \end{bmatrix} dx \wedge dy) = \operatorname{res}(h, (-\frac{\partial}{\partial y}) \begin{bmatrix} 1 \\ xy \end{bmatrix} dx \wedge dy)$$

$$= \operatorname{res}(\frac{\partial h}{\partial y}, \begin{bmatrix} 1 \\ xy \end{bmatrix} dx \wedge dy)$$

Notice that

$\frac{\partial}{\partial y}$ is the formal adjoint of $(-\frac{\partial}{\partial y})$

## contents

# local duality and differential operators

Let

$K[x] = K[x_1, x_2, \ldots, x_n]$: polynomial ring $K = \mathbb{Q}$

$\mathfrak{p} = (x_1, x_2, \ldots, x_n) \subset K[x]$: prime ideal

$\mathfrak{q}$: a $\mathfrak{p}$-primary ideal

Let $H^n_{\mathfrak{p}}(K[x])$ denote the algebraic local cohomology defined by

$$H^n_{\mathfrak{p}}(K[x]) = \lim_{k\to\infty} \mathrm{Ext}^n_{K[x]}(K[x]/\mathfrak{p}^k, K[x]).$$

Let $\delta_{\mathfrak{p}}$ denote the local cohomology class defined by

$$\delta_{\mathfrak{p}} = \left[ \frac{1}{x_1 x_2 \cdots x_n} \right] \in H^n_{\mathfrak{p}}(K[x])$$

Let $D = K[x, \frac{\partial}{\partial x}]$: the Weyl algebra

**Lemma** $H^n_{\mathfrak{p}}(K[x]) = D\delta_{\mathfrak{p}}$.

The local cohomology group $H^n_{\mathfrak{p}}(K[x])$ can be generated by $\delta_{\mathfrak{p}}$ as a D-module:

Let

$$H_{\mathfrak{p}} = \{\psi \in H^n_{\mathfrak{p}}(K[x]) \mid p\psi = 0, \ \forall p \in \mathfrak{p}\},$$

$$H_{\mathfrak{q}} = \{\psi \in H^n_{\mathfrak{p}}(K[x]) \mid q\psi = 0, \ \forall q \in \mathfrak{q}\}$$

we have

$$H_{\mathfrak{p}} = \mathrm{Span}_K\{\delta_{\mathfrak{p}}\}$$

# local duality and differential operators

Let

$$H_{\mathfrak{q}} = \text{Span}_K\{\psi_1, \psi_2, \cdots, \psi_m\}, \;\; m = \dim_K(H_{\mathfrak{q}})$$

$$\psi_j = R_j \delta_{\mathfrak{p}}, \;\; R_j \in D, j = 1, 2, \cdots, m$$

$$L_j = R_j^*\text{: the formal adjoint of } R_j, \;\; j = 1, 2, \cdots, m$$

**Theorem** Let $h \in K[x]$. Then $h \in \mathfrak{q}$ iff

$$L_j h \in \mathfrak{p} = (x_1, x_2, \cdots, x_n), \;\; j = 1, 2, \cdots, m$$

proof

$$\begin{aligned} \text{res}(h, \psi_j dx) &= \text{res}(h, (R_j \delta_{\mathfrak{p}}) dx) \\ &= \text{res}(L_j h, \delta_{\mathfrak{p}} dx) = (L_j h)(0) \end{aligned}$$

## contents

# zero-dimensional case

Let

$\mathfrak{p} \subset K[x] = K[x_1, x_2, \ldots, x_n]$: a zero-dimensonal prime ideal

$H^n_{\mathfrak{p}}(K[x]) = \lim_{k\to\infty} \mathrm{Ext}^n_{K[x]}(K[x]/\mathfrak{p}^k, K[x])$

Let

$\{p_1, p_2, \ldots, p_n\}$: a set of generators of the ideal $\mathfrak{p}$

$$\delta_{\mathfrak{p}} = \det\left(\frac{\partial(p_1, p_2, \ldots, p_n)}{\partial(x_1, x_2, \ldots, x_n)}\right)\left[\begin{array}{c} 1 \\ p_1 p_2 \cdots p_n \end{array}\right] \in H^n_{\mathfrak{p}}(K[x]).$$

$H_{\mathfrak{p}} = \{\psi \in H^n_{\mathfrak{p}}(K[x]) \mid p\psi = 0,\ \forall p \in \mathfrak{p}\}$

Then,

$H_{\mathfrak{p}} = \{h(x)\delta_{\mathfrak{p}} \mid h(x) \in K[x]/\mathfrak{p}\}$

## zero-dimensional case

Let $\mathfrak{q}$: a $\mathfrak{p}$-primary ideal

$$H_{\mathfrak{q}} = \{\psi \in H^n_{\mathfrak{p}}(K[x]) \mid q\psi = 0,\ \forall q \in \mathfrak{q}\}$$

Notice that

$H_{\mathfrak{p}}, H_{\mathfrak{q}}$ have the structure of a vector space over the field $K[x]/\mathfrak{p}$

**Lemma**

$$\mathrm{Hom}_{K[x]}(K[x]/\mathfrak{p}, H_{\mathfrak{p}}(K[x])) = H_{\mathfrak{p}}$$

$$\mathrm{Hom}_{K[x]}(K[x]/\mathfrak{q}, H_{\mathfrak{p}}(K[x])) = H_{\mathfrak{q}}$$

Let $D = K[x, \frac{\partial}{\partial x}]$: the Weyl algebra

$H^n_{\mathfrak{p}}(K[x])$: holonomic D-module

**Lemma** $H^n_{\mathfrak{p}}(K[x]) = D\delta_{\mathfrak{p}}$

# zero-dimensional case

Now, we introduce two D-modules $M_{\mathfrak{p}}, M_{\mathfrak{q}}$ as

$M_{\mathfrak{p}} = D/D\mathfrak{p},\ M_{\mathfrak{q}} = D/D\mathfrak{q}$: holonomic D-modules

Consider

$\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$: the set of D-linear homomorphisms between the two left D-modules

$\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$: Noetherian space of $H_{\mathfrak{q}}$

(i) a finite dimensional vector space over $K$.

(ii) a finite dimensional vector space over the fields $K[x]/\mathfrak{p}$.

Note

$$\dim_K(H_{\mathfrak{q}}) = \dim_{K[x]/\mathfrak{p}}(\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})) \cdot \dim_K(H_{\mathfrak{p}})$$

**Proposition**

(i) $\mathrm{Hom}_D(M_{\mathfrak{p}}, H^n_{\mathfrak{p}}(K[x])) = H_{\mathfrak{p}}$

(ii) $\mathrm{Hom}_D(M_{\mathfrak{q}}, H^n_{\mathfrak{p}}(K[x])) = H_{\mathfrak{q}}$

From

$$\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}}) \times \mathrm{Hom}_D(M_{\mathfrak{p}}, H^n_{\mathfrak{p}}(K[x])) \to \mathrm{Hom}_D(M_{\mathfrak{q}}, H^n_{\mathfrak{p}}(K[x]))$$

we have

$$\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}}) \times H_{\mathfrak{p}} \to H_{\mathfrak{q}} \ (surjective)$$

Let

$\rho \in \mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$ and $1 \in M_{\mathfrak{q}} = D/D\mathfrak{q}$. Then the image

$\rho(1) \in M_{\mathfrak{p}} = D/D\mathfrak{p}$ can be represented by

a partial differential operator, say $R$, in $D$

**Lemma** Let $R \in D$. Then,

$R$ is a representative of an element of $\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$, iff $R$ satisfies

$qR \in D\mathfrak{p}, \ \forall q \in \mathfrak{q}$.

$\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$ is computable!!

Let $\{\rho_1, \rho_2, \ldots, \rho_m\}$: a basis, as a vector space over $K[x]/\mathfrak{p}$,

of the Noetherian space $\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})$

$(m = \dim_{K[x]/\mathfrak{p}}(\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}})))$

$R_j \in D$: a representative of $\rho_j$ $(j = 1, 2, \ldots, m)$.

Since $\mathrm{Hom}_D(M_{\mathfrak{q}}, M_{\mathfrak{p}}) \times H_{\mathfrak{p}} \to H_{\mathfrak{q}}$ is surjective

any alement $\psi \in H_{\mathfrak{q}}$ can be represented as

$\psi = \sum_j R_j(b_j \delta_{\mathfrak{p}}), \ b_j \in K[x]/\mathfrak{p}$

# zero-dimensional case

Let

$L_j = R_j^*$: the formal adjoint of $R_j$, $j = 1, 2, \ldots, m$

**Theorem**

$\mathfrak{q} \subset K[x]$: a $\mathfrak{p}$-primary ideal. Let $h \in K[x]$. Then

$h \in \mathfrak{q}$, iff $L_j h \in \mathfrak{p}$, $j = 1, 2, \ldots, m$

$\{L_1, L_2, \cdots, L_m\}$ constructed from $\mathrm{Hom}_D(M_\mathfrak{q}, M_\mathfrak{p})$ completely describes the multiplicity structure of the primaly ideal $\mathfrak{q}$

$\{L_1, L_2, \cdots, L_m\}$ gives rise to the Noetherian differential operators in the sence of Ehrenpreis-Palamodov

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2. local duality and differential operators
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   - definition
   - algorithm
4. positive dimensional case
5. duality

## zero-dimensional case

Let

$I \subset K[x]$; zero-dimensional ideal, $\sqrt{I}$: radical

$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_\ell$: primary decomposition,

$\mathfrak{p}_k \subset K[x]$: associated prime of $\mathfrak{q}_k$, $k = 1, 2, \cdots, \ell$

$\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_\ell$

Let

$M_I = D/DI$: holonomic D-module

$M_{\mathfrak{p}_k} = D/D\mathfrak{p}_k, \; M_{\mathfrak{q}_k} = D/D\mathfrak{q}_k, \; k = 1, 2, \cdots, \ell$

**Proposition**

$$\mathrm{Hom}_D(M_I, M_{\mathfrak{p}_k}) = \mathrm{Hom}_D(M_{\mathfrak{q}_k}, M_{\mathfrak{p}_k})$$

**corollary** Let $R \in D$. Then,

$R$ is a representative of an element of $\mathrm{Hom}_D(M_{\mathfrak{q}_k}, M_{\mathfrak{p}_k})$, iff $R$ satisfies

$$gR \in D\mathfrak{p}_k, \ \forall g \in I.$$

$\mathrm{Hom}_D(M_{\mathfrak{q}_k}, M_{\mathfrak{p}_k})$ is computable from

(i) a set of generators of the ideal $I$

(ii) a Groebner basis of the prime component $\mathfrak{p}_k$

S. T. 1997-2002, Grothendieck duality and basic concepts

S. T. 2003-2004, algorithm for computing Noetherian differential operators

S. T. 2004, algorithm for computing Grothendieck local residues

# zero-dimensional case

Implementation and applications

K. Ohara and S.T.

An algorithm for computing Grothendieck local residues II

Mathematics in Computer Scinece **14** (2020), 483-496

K. Nabeshima and S. T.

Effective algorithms for computing Noetherian representations of zero-dimensional ideals,

Applicable Algebra in Engineering, Computation and Computing **33** (2022), 867-899

# zero-dimensional case

**Example** $I = ((x^2+y^2)^2 + 3x^2y - y^3, x^2+y^2-1) \subset K[x,y]$

$I = \mathfrak{q}_0 \cap \mathfrak{q}_1$: primary decomposition

$\mathfrak{q}_0 = (x^2, y-1)$, $\mathfrak{q}_1 = ((4x^2-3)-2(2y+1), (2y+1)^2)$

$\mathfrak{p}_0 = (x, y-1)$, $\mathfrak{p}_1 = (4x^2-3, 2y+1)$: associated primes

local cohomology and Noetherian operators

$$H_{\mathfrak{p}_0} = \mathrm{Span}_K(\begin{bmatrix} 1 \\ (x)(y-1) \end{bmatrix}),$$

$$H_{\mathfrak{q}_0} = \mathrm{Span}_K(\begin{bmatrix} 1 \\ (x)(y-1) \end{bmatrix}, \begin{bmatrix} 1 \\ (x^2)(y-1) \end{bmatrix})$$

$NT_{\mathfrak{q}_0} = \{1, (-\frac{\partial}{\partial x})\}$: Noetherian differential operators

# zero-dimensional case

Noetherian operators

$H_{\mathfrak{p}_1} \subset H_{\mathfrak{q}_1}$: $K[x,y]/\mathfrak{p}_1$-vector spaces

$$H_{\mathfrak{p}_1} = \mathrm{Span}_{K[x,y]/\mathfrak{p}_1}\left(\begin{bmatrix} 1 \\ (4x^2-3)(2y+1) \end{bmatrix}\right),$$

$$H_{\mathfrak{q}_1} = \mathrm{Span}_{K[x,y]/\mathfrak{p}_1}\left(\begin{bmatrix} 1 \\ (4x^2-3)(2y+1) \end{bmatrix}, \begin{bmatrix} 2 \\ (4x^2-3)^2(2y+1) \end{bmatrix} + \begin{bmatrix} 1 \\ (4x^2-3)(2y+1)^2 \end{bmatrix}\right)$$

$NT_{\mathfrak{q}_1} = \{1, (-\frac{\partial}{\partial x}) + (-\frac{\partial}{\partial y})2x\}$: Noetherian differential operators

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1. Introduction
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3. zero-dimensional case
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    - isolated components
    - embedded components
5. duality

# positive dimensional case

$\mathfrak{q}$: primary ideal in $K[x,y]$, $\mathfrak{p} := \sqrt{\mathfrak{q}}$: the associated prime of $\mathfrak{q}$

$x = (x_1, x_2, \cdots, x_d), y = (y_{d+1}, y_{d+2}, \cdots, y_n)$

$\{x\}$: max. indep. set of $\mathfrak{p}$, $d$: the dimension of $\mathfrak{p}$

$\mathfrak{q}^e, \mathfrak{p}^e \subset K(x)[y]$: zero-dimensional ideals in $K(x)[y] = K[x,y]^e$

Let

$D^e := K(x)[y, \frac{\partial}{\partial y}]$, $M_{\mathfrak{q}^e} = D^e/D^e\mathfrak{q}^e$, $M_{\mathfrak{p}^e} = D^e/D^e\mathfrak{p}^e$

consider

$\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$ as a vector space over $K(x)[y]/\mathfrak{p}^e$

$m := \dim_{K(x)[y]/\mathfrak{p}}(\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e}))$: multiplicity of $\mathfrak{q}$ over $\mathfrak{p}$

$\{\rho_1, \rho_2, \cdots, \rho_m\}$: a basis of $\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$

Noetherian operators for local cohomology

$NT = \{R_1, R_2, \cdots, R_m\}$: their representatives in $D$

# positive dimensionall cace

**Example**

$\mathfrak{p} = (x_2x_0^2 - x_1^3, x_2^2x_0 - x_3x_1^2, x_3x_0 - x_2x_1, x_3^2x_1 - x_2^3)$: prime ideal

(W. Vogel, 1984)

$\mathfrak{p}^e = (x_3^3x_o - x_2^4, x_3^2x_1 - x_2^3) \subset K(x_2, x_3)[x_0, x_1]$: zero-dimensional

Let

$$I = ((x_2x_0^2 - x_1^3)^2, (x_2^2x_0 - x_3x_1^2)^2, (x_3x_0 - x_2x_1)^2, x_3^2x_1 - x_2^3)$$

$I = \mathfrak{q}_0 \cap \mathfrak{q}_1 \cap \mathfrak{q}_2$: primary decomposition,

$$\sqrt{\mathfrak{q}_0} = (x_0, x_1, x_2, x_3), \sqrt{\mathfrak{q}_1} = (x_1, x_2, x_3), \sqrt{\mathfrak{q}_2} = \mathfrak{p}$$

$$\mathfrak{q}_2 = (x_1^4 - 3x_2x_1x_2^2 + 2x_3x_0^3, x_2x_1^3 - 2x_3x_1^2x_0 + x_2^2x_0^2, x_3x_1^3 - 2x_2^2x_1x_0 + x_2x_3x_0^2, x_2^2x_1^2 - 2x_2x_3x_1x_0 + x_3^2x_0^2, x_3^2x_1 - x_2^3)$$

$$\mathfrak{q}_2^e = ((x_3^3x_0 - x_2^4)^2, x_3^2x_1 - x_2^3) \subset K(x_2, x_3)[x_0, x_1]$$

## positive dimensional case

Noetehrian operator NT of $\mathfrak{q}_2$

From

$$\mathfrak{q}_2^e = ((x_3^3 x_0 - x_2^4)^2, x_3^2 x_1 - x_2^3),\ \mathfrak{p}^e = (x_3^3 x_o - x_2^4, x_3^2 x_1 - x_2^3),$$

we have

$$\mathrm{NT} = \{1, (-\frac{\partial}{\partial x_0})\}$$

**Membership problem** for this case

Let $h \in K[x_0, x_1, x_2, x_3]$. then

$h \in \mathfrak{q}_2$ if and only if

$$h \in \mathfrak{p}^e \text{ and } \frac{\partial h}{\partial x_0} \in \mathfrak{p}^e \text{ in } K(x_2, x_3)[x_0, x_1]$$

Notice that

Groebner basis of $\mathfrak{q}_2$ consists of 5 elements.
Groebner basis of $\mathfrak{p}^e = \sqrt{\mathfrak{q}_2^e}$ consists of 2 elements.

# positive dimensional case

**summing-up**

$\mathfrak{p} \subset K[x, y]$: prime ideal,

$x = (x_1, x_2, \cdots, x_d), y = (y_{d+1}, y_{d+2}, \cdots, y_n)$

$\{x\}$: max. indep. set of $\mathfrak{p}$,

$\mathfrak{p}^e \subset K(x)[y]$: the extension of $\mathfrak{p}$

$$H^n_{\mathfrak{p}^e}(K(x)[y]) = \lim_{k\to\infty} \mathrm{Ext}^{n-d}_{K(x)[y]}(K(x)[y]/(\mathfrak{p}^e)^k, K(x)[y])$$

$\mathfrak{q}$: $\mathfrak{p}$-primary ideal in $K[x, y]$

$\mathfrak{q}^e \subset K(x)[y] = K[x, y]^e$

Let

$$H_{\mathfrak{p}^e} = \{\psi \in H^n_{\mathfrak{p}^e}(K(x)[y]) \mid \mathfrak{p}^e\psi = 0\}$$

$$H_{\mathfrak{q}^e} = \{\psi \in H^n_{\mathfrak{p}^e}(K(x)[y]) \mid \mathfrak{q}^e\psi = 0\}$$

# positive dimensional case

Let

$D^e := K(x)[y, \frac{\partial}{\partial y}]$, $M_{\mathfrak{q}^e} = D^e/D^e\mathfrak{q}^e$, $M_{\mathfrak{p}^e} = D^e/D^e\mathfrak{p}^e$

$\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$: a vector space over $K(x)[y]/\mathfrak{p}^e$

$m := \dim_{K(x)[y]/\mathfrak{p}}(\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e}))$: multiplicity of $\mathfrak{q}$ over $\mathfrak{p}$

$\{\rho_1, \rho_2, \cdots, \rho_m\}$: a basis of $\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$

Noetherian operators for local cohomology classes

$NT = \{R_1, R_2, \cdots, R_m\}$: their representatives in $D^e$

Since $\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e}) \times H_{\mathfrak{p}^e} \to H_{\mathfrak{q}^e}$ is surjective

any alement $\psi \in H_{\mathfrak{q}^e}$ can be represented as

$\psi = \sum_j R_j(b_j\delta_{\mathfrak{p}})$, $b_j \in K(x)[y]/\mathfrak{p}^e$

## positive dimensional case

$NT = \{R_1, R_2, \cdots, R_m\}$: representatives in $D^e$ of the basis

$\{\rho_1, \rho_2, \cdots, \rho_m\}$, over $K(x)[y]/\mathfrak{p}^e$, of $\mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$

Let

$\{L_1, L_2, \cdots, L_m\}$: the set of formal adjoints of NT

$h \in K[x, y]$

Then, $h \in \mathfrak{q}$, iff

$L_j h \in \mathfrak{p}^e$, $j = 1, 2, \cdots, m$

Let

$I \subset K[x, y]$: ideal

$x = (x_1, x_2, \cdots, x_d), y = (y_{d+1}, y_{d+2}, \cdots, y_n)$

$\{x\}$: max. indep. set of $I$,

# positive dimensional case

Let

$\mathfrak{p}^e$: a prime component of $\sqrt{I^e} \subset K(x)[y]$

$\mathfrak{q}^e$: $\mathfrak{p}^e$-primary component of $I^e \subset K(x)[y]$

$(\mathfrak{p} = \mathfrak{p}^e \cap K[x, y], \ \mathfrak{q} = \mathfrak{q}^e \cap K[x, y])$

Let

$M_{I^e} = D^e / D^e I^e$. Then

**Proposition**

$$\mathrm{Hom}_{D^e}(M_{I^e}, M_{\mathfrak{p}^e}) = \mathrm{Hom}_{D^e}(M_{\mathfrak{q}^e}, M_{\mathfrak{p}^e})$$

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## contents

**Primary decomposition via Noether operators**

$I = \langle f_1^2, f_2, f_3, f_4 \rangle \subset K[x, y, z, w]$

$f_1 = x^2 z - y^3, f_2 = xz^2 - y^2 w, f_3 = xw - yz, f_4 = yw^2 - z^3$

$I = \mathfrak{q}_2 \cap \mathfrak{q}_1,$

$\mathfrak{q}_2 = (f_1, f_2, f_3, f_4)$: prime ideal

$\mathfrak{q}_1 = (y^6, yz, z^2, w)$, embedded component (not unique)

$(\mathfrak{q}_2)^e = (x^2 z - y^3, x^3 w - y^4) \subset K(x, y)[z, w]$: prime ideal

$\left[ \dfrac{1}{(z - \frac{y^3}{x^2})\ (w - \frac{y^4}{x^3})} \right]$: local cohomology class

Assume that

$\mathfrak{p}_1 = (\sqrt{\mathfrak{q}_1}) = (y, z, w)$ is given, (uniquely determined)

# positive dimensional case

computation on $V(y,z,w)$ of $q_1$

$$\tau_k = \left[ \begin{matrix} 1 \\ (y^k)\ (z - \frac{y^3}{x^2})\ (w - \frac{y^4}{x^3}) \end{matrix} \right], \ \ k = 1, 2, 3, \cdots$$

$$f_1^2\tau_k = f_2\tau_k = f_3\tau_k = f_4\tau_k = 0, \ \ k = 1, 2, 3, \cdots$$

$$\tau_1 = \left[ \begin{matrix} 1 \\ yzw \end{matrix} \right], \tau_2 = \left[ \begin{matrix} 1 \\ y^2zw \end{matrix} \right], \tau_3 = \left[ \begin{matrix} 1 \\ y^3zw \end{matrix} \right]$$

$$\tau_4 = \left[ \begin{matrix} 1 \\ y^4zw \end{matrix} \right] + \frac{1}{x^2}\left[ \begin{matrix} 1 \\ yz^2w \end{matrix} \right],$$

$$\tau_5 = \left[ \begin{matrix} 1 \\ y^5zw \end{matrix} \right] + \frac{1}{x^2}\left[ \begin{matrix} 1 \\ y^2z^2w \end{matrix} \right] + \frac{1}{x^3}\left[ \begin{matrix} 1 \\ yzw^2 \end{matrix} \right]$$

$$\tau_6 = \left[ \begin{matrix} 1 \\ y^6zw \end{matrix} \right] + \frac{1}{x^2}\left[ \begin{matrix} 1 \\ y^3z^2w \end{matrix} \right] + \frac{1}{x^3}\left[ \begin{matrix} 1 \\ y^2zw^2 \end{matrix} \right]$$

$$\tau_7 = \left[ \begin{matrix} 1 \\ y^7zw \end{matrix} \right] + \frac{1}{x^2}\left[ \begin{matrix} 1 \\ y^4z^2w \end{matrix} \right] + \frac{1}{x^3}\left[ \begin{matrix} 1 \\ y^3zw^2 \end{matrix} \right] + \frac{1}{x^4}\left[ \begin{matrix} 1 \\ yz^3w \end{matrix} \right]$$

## positive dimensional case

compute local cohomology $\sigma$ s.t.

(i) $f_1^2\sigma = f_2\sigma = f_3\sigma = f_4\sigma = 0$

(ii) $\sigma \notin \mathrm{Span}\{\tau_k, k = 1, 2, 3, \cdots\}$

Result

$$\sigma_1 = [\frac{1}{yz^2w}],\ \sigma_2 = [\frac{1}{y^2z^2w}] + \frac{1}{x}[\frac{1}{yzw^2}],$$

$$\sigma_3 = [\frac{1}{y^3z^2w}] + \frac{1}{x}[\frac{1}{y^2zw^2}]$$

$$\tau_4 = [\frac{1}{y^4zw}] + \frac{1}{x^2}\sigma_1$$

$$\tau_5 = [\frac{1}{y^5zw}] + \frac{1}{x^2}\sigma_2$$

$$\tau_6 = [\frac{1}{y^6zw}] + \frac{1}{x^2}\sigma_3$$

local cohomology classes that define primary componet supported on $V(y, z, w)$

$$\tau_1 = \left[\frac{1}{yzw}\right],\ \tau_2 = \left[\frac{1}{y^2zw}\right],\ \tau_3 = \left[\frac{1}{y^3zw}\right]$$

$$\tau'_4 = \left[\frac{1}{y^4zw}\right],\ \tau'_5 = \left[\frac{1}{y^5zw}\right],\ \tau'_6 = \left[\frac{1}{y^6zw}\right]$$

$$\sigma_1 = \left[\frac{1}{yz^2w}\right],\ \sigma_2 = \left[\frac{1}{y^2z^2w}\right] + \frac{1}{x}\left[\frac{1}{yzw^2}\right],$$

$$\sigma_3 = \left[\frac{1}{y^3z^2w}\right] + \frac{1}{x}\left[\frac{1}{y^2zw^2}\right]$$

## positive dimensional case

Example ($J_{3,\infty}$) $f(x,y) = y^3 + x^3y^2$

$J = (f, x^2y^2, 3y^2 + 2x^3y) \subset K[x,y]$: Jacobi ideal

$J = \mathfrak{q}_1 \cap \mathfrak{q}_0$, $\mathfrak{q}_0$: embedded component,

$J^e = \mathfrak{q}_1^e = (y)$, $R_0 = 1$; Noetherian differential operator of $J^e$

$\mathfrak{p}_0 = \sqrt{\mathfrak{q}_0}$: the associated prime

compute Noether operators of $J$ with respect to $\mathfrak{p}_0 = (x,y)$

Let

$T^k = (-\frac{\partial}{\partial x})^k$, $k = 0, 1, 2, .....$

from $x^2y^2R_0T^k = 0$, $(3y^2 + x^3y^2)R_0T^k = 0 \bmod D(x,y)$

we have

$$J(R_0T^k[\frac{1}{xy}]) = 0, k = 0, 1, 2, .....$$

# positive dimensional case

compute differential operators $S$ mod $D(x, y)$ s.t.

(i) $S \notin \mathrm{Span}_K\{T^k \mid k = 0, 1, 2, \ldots\}$

(ii) $JS \in D(x, y)$

$$S_1 = (-\tfrac{\partial}{\partial y}),\ S_2 = (-\tfrac{\partial}{\partial x})(-\tfrac{\partial}{\partial y}),\ S_3 = (-\tfrac{\partial}{\partial x})^2(-\tfrac{\partial}{\partial y}),$$

$$S_4 = (-\tfrac{\partial}{\partial y})^2 - \tfrac{1}{2}(-\tfrac{\partial}{\partial x})^3(-\tfrac{\partial}{\partial y}),$$

$$S_5 = (-\tfrac{\partial}{\partial x})(-\tfrac{\partial}{\partial y})^2 - \tfrac{1}{2}(-\tfrac{\partial}{\partial x})^4(-\tfrac{\partial}{\partial y})$$

from these dada, one can compute primary ideal $\mathfrak{q}_0$

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## duality

L. Ehrenpreis (1961, 1970)

V. Palamodov (1967)

J. E. Bjork (1978)

L. Hormander (1990)

U. Oberst (1999)

D. Zeilberger (2000)

A. Damiano, I. Sabadini and D. C. Struppa (2007)

I. Nonkań (2013)

Y. Cid-Ruiz (2020)

Y. Cid-Ruiz, R. Homes and B. Strumfels (2021)

Y. Cid-Ruiz, J. Chen, M. Harkonen, R. Krone and A. Leykin (2022)

# duality

田島慎一, 代数的局所コホモロジー類のローラン展開と L. Ehrenpreis の Noether 作用素, 京都大学数理解析研究所講究録 **1138** (2000), 87-95

S. Tajima, An algorithm for computing the Noetherian operator representations and its applications to constant coefficients holonomic PDE's, Tools for Mathematical Modellings (2001), St. Peterbourg 154-160

S. T.ajima, Exponential polynomial solutions and the Fourier-Borel transforms of algebraic local cohomology classes, in Mathematical Analysis and Complex Analysis, World Scientific (2002), 284-296

田島慎一, 確定特異点型ホロノミック系の零次元代数的局所コホモロジー解, 京都大学数理解析研究所講究録 **1236** (2003), 121-132

田島慎一, 零次元イデアルと Noether 作用素アルゴリズム, 京都大学数理解析研究所講究録 **1395** (2004), 57-63

田島慎一, Noether 作用素と多変数留数計算アルゴリズム, 京都大学数理解析研究所講究録 **1431** (2005), 123-136

# duality

田島慎一, 一変数留数計算アルゴリズムについて, 京都大学数理解析研究所講究録 **1509** (2006), 24-50

S. Tajima, On Noetherian differential operators attached to a zero-dimensional primary ideal -shape basis case-, Proc. 12th Internat. Conf. on Finite or Infinite Dimensional Complex Analysis, 357-366, Kyushu Univ. Press (2005)

S. Tajima, Y. Nakamura and K. Nabeshima, Standard bases and algebraic local cohomology for zero dimensional ideals, Adv. Studies in Pure Math. **56** (2009), 341-361.

K. Ohara and S.Tajima, An algorithm for computing Grothendieck local residues II, Mathematics in Computer Scinece **14** (2020), 483-496

鍋島克輔, 田島慎一, 零次元準素イデアルのネター作用素の計算と応用, 京都大学数理解析研究所講究録 **2185** (2021), 1-15

K. Nabeshima and S. Tajima, Effective algorithms for computing Noetherian representations of zero-dimensional ideals, Applicable Algebra in Engineering, Computation and Computing **33** (2022), 867-899

深作亮也, 田島慎一, 一変数留数計算について, 京都大学数理解析研究所講究録 **2224** (2022), 70-78

鍋島克輔, 田島慎一, 正次元イデアルのネター作用素の計算と応用, 京都大学数理解析研究所講究録 **2255** (2023), 75-87

K. Nabeshima and S. Tajima, Effective algorithm for computing Noetherian operators of positive dimensional ideals, Lecture Notes in Computer Science **14139** (2023), 272-291

田島慎一, Local cohomology に対するネター作用素とホロノミー D-加群, 京都大学数理解析研究所講究録 **2280** (2024), 176-185

S. Tajima. Y. Umeta and K. Nabeshima, Noetherian operators for local cohomology classes and holonomic D-modules associated with non-isolated hypersurface singularities, submitted.

Thank you very much for your attention

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