EXTENDED CANONICAL IDEALS AND GOTO RINGS

NAOKI ENDO

Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$ admitting a canonical module K_A. We assume A contains a canonical ideal I, i.e., I is an ideal of A such that $I \neq A$ and $I \cong \mathrm{K}_A$ as an A-module. Introduced by Northcott and Rees, for ideals J and Q of A with $Q \subseteq J$, we say that Q is a *reduction* of J if $J^{r+1} = QJ^r$ for some $r \geq 0$; the least such integer r is called the *reduction number* of J with respect to Q. An ideal J is called an *extended canonical ideal* of A if $J = I + Q$ for some parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ satisfying that $a_1 \in I$ and Q is a reduction of J. Note that $J\overline{A}$ forms a canonical ideal of $\overline{A} = A/\mathfrak{q}$ if $d \geq 2$, where $\mathfrak{q} = (a_2, \ldots, a_d)$.

The aim of this talk is, as part of stratification of Cohen-Macaulay rings, to introduce the notion of Goto rings, generalizing the notion of almost Gorenstein rings defined by Barucci and Fröberg [1] for one-dimensional analytically unramified local rings; Goto, Matsuoka, and Phuong [2] for one-dimensional Cohen-Macaulay local rings; and Goto, Takahashi, and the speaker of this talk [3] for Cohen-Macaulay graded/local rings of arbitrary dimension. What has dominated the series of researches on almost Gorenstein rings is the fact that the reduction numbers of extended canonical ideals are at most 2; we define Goto rings as Cohen-Macaulay rings admitting such extended canonical ideals.

REFERENCES

[1] V. Barucci and R. Fröberg, *One-dimensional almost Gorenstein rings*, J. Algebra, **188** (1997), no. 2, 418–442.

[2] S. Goto, N. Matsuoka, and T. T. Phuong, *Almost Gorenstein rings*, J. Algebra, **379** (2013), 355–381.

[3] S. Goto, R. Takahashi, and N. Taniguchi, *Almost Gorenstein rings -towards a theory of higher dimension*, J. Pure Appl. Algebra, **219** (2015), 2666–2712.

SCHOOL OF POLITICAL SCIENCE AND ECONOMICS, MEIJI UNIVERSITY, 1-9-1 EIFUKU, SUGINAMI-KU, TOKYO 168-8555, JAPAN

Email address: endo@meiji.ac.jp

BEST POSSIBLE DEGREE BOUND FOR GRÖBNER BASES OF 1-DIMENSIONAL IDEALS

KOICHIRO TANI

Let $\Bbbk$ be a field, $S = \Bbbk[x_1, \ldots, x_n]$ a polynomial ring. A *monomial order* $\preceq$ is a total order on the set of all monomials in S such that $u \preceq v$ implies $uw \preceq vw$ for each monomials u, v, w and 1 is the least monomial. The largest term in $f \in S$ with respect to $\preceq$ is called the *initial term* of f and we write $\mathrm{in}_{\preceq}(f)$. A $\Bbbk$-vector space spanned by $\{\mathrm{in}_{\preceq}(f) \mid f \in I\}$ is the *initial ideal* of I and write $\mathrm{in}_{\preceq}(I)$. We call a set of polynomials $\{g_1, \ldots, g_t\}$ *Gröbner basis* if $\mathrm{in}_{\preceq}(I) = (\mathrm{in}_{\preceq}(g_1), \ldots, \mathrm{in}_{\preceq}(g_t))$. A Gröbner basis $\{g_1, \ldots, g_t\}$ is *reduced* if they are monic and for each g_i, there is no monomial in g_i divisible by $\mathrm{in}_{\preceq}(g_j)$ for any $j \neq i$. In this talk, we will consider the next question.

Question 1. *Let* $\max.\mathrm{GB}.\deg_{\preceq}(I)$ *be the maximal degree of reduced Gröbner basis and* $\deg(I)$ *the maximal degree of minimal generators. How does* $\max.\mathrm{GB}.\deg_{\preceq}(I) - \deg(I)$ *become large?*

From now on, let I be a homogeneous ideal of S. The Hilbert series of S/I is $\mathrm{HS}(S/I;t) \coloneqq \sum_{i=0}^{\infty} \dim_{\Bbbk}(S/I)_i t^i$. For all I and $\preceq$, $\mathrm{HS}(S/I;t) = \mathrm{HS}(S/\mathrm{in}_{\preceq}(I))$. This fact is known as Macaulay's Theorem. Furthermore, by the definition of Gröbner basis, $\max.\mathrm{GB}.\deg_{\preceq}(I) = \deg(\mathrm{in}_{\preceq}(I))$. The degree $\deg(\mathrm{in}_{\preceq}(I))$ can be bounded by the degree of the lexsegment ideal.

Definition 2. Let a term order $\preceq$ be the lexicographic order. A monomial ideal J is called the *lexsegment ideal* if, for all monomials $m \in J$, any monomials m' satisfying $\deg(m) = \deg(m')$ and $m \preceq m'$ also belong to J.

Theorem 3 (Macaulay's Theorem)**.** *For any homogeneous ideal I, there uniquely exists a lexsegment ideal J such that* $\mathrm{HS}(S/I;t) = \mathrm{HS}(S/J;t)$*. Moreover, for any $d \geq 0$, J has at least as many generators in degree d as any other monomial ideal with the same Hilbert series.*

Corollary 4. *For any homogeneous ideal I, let J be the lexsegment ideal of* $\mathrm{HS}(S/I;t)$*. Then, we have* $\max.\mathrm{GB}.\deg_{\preceq}(I) \leq \deg(J)$ *for any monomial order $\preceq$.*

We want to determine whether the bound $\deg(J)$ is the best possible. It is sufficient to find an ideal as described in the following question.

Question 5. *Find any homogeneous ideal I and monomial order $\preceq$ where $\deg(I)$ is as small as possible and $\mathrm{in}_{\preceq}(I)$ is equal to the lexsegment ideal of $\mathrm{HS}(S/I;t)$.*

For simplicity, let I be generated by regular sequence. We set parameterized polynomials

$$\begin{aligned} f_1 &= a_{1,1}x_1^{d_1} + a_{1,2}x_1^{d_1-1}x_2 + \cdots + a_{1,r_1}x_n^{d_1} \\ &\vdots \\ f_s &= a_{s,1}x_1^{d_s} + a_{s,2}x_1^{d_s-1}x_2 + \cdots + a_{s,r_s}x_n^{d_s} \end{aligned}$$

where $r_i = \binom{n+d_i-1}{d_i}$. Let $N = \sum_{i=1}^{s} r_i$. Choosing a point in $\Bbbk^N$ and assigning each entry to each parameter $a_{i,j}$ is equivalent to determining specific polynomials $f_1, \ldots, f_s \in S$ as generators of I. We have the following lemma, referring to discussions about generic initial ideals with exterior algebra. (see [1, Section 15.9]).

Lemma 6. *Fix a monomial order $\preceq$. There is a Zariski open dense set $U \subset \Bbbk^N$ such that for all $\mathbf{a} \in U$, initial ideals of I generated by $f_1, \ldots, f_s$ corresponding to $\mathbf{a}$ are the same.*

As a temporary term, we call the initial ideal above the initial ideal of *almost case*.

Theorem 7. *If $n = 4, s = 2, d_1 = d_2 = 2$, there is no regular sequence whose initial ideal with respect to lexicographic order is the lexsegment ideal.*

Theorem 8. *If $n = 3, s = 2, 2 \leq d_1, d_2 \leq 3$, or if $n = 4, s = 3, 2 \leq d_1, d_2, d_3 \leq 3$, then initial ideals of almost case with respect to lexicographic order are lexsegment ideals.*

We can expect the following claims. If the Krull dimension of S/I is 2, then the initial ideals of I are never lexsegment ideals, but if the Krull dimension of S/I is 1, then the initial ideals of almost case of I with respect to lexicographic order are lexsegment ideals.

References

[1] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Osaka, Japan

Email address: `tani-k@ist.osaka-u.ac.jp`

BINOMIAL EDGE RINGS OF COMPLETE BIPARTITE GRAPHS

AKIHIRO HIGASHITANI

1. Binomial edge rings of finite simple graphs

Let G be a graph on the vertex set $V(G) = [d] := \{1, \dots, d\}$ with the edge set $E(G)$. Let $\Bbbk$ be a field and let $S = \Bbbk[x_1, \dots, x_d, y_1, \dots, y_d]$. We introduce a subring $\mathcal{R}(G)$ of S as follows:

$$\mathcal{R}(G) := \Bbbk[x_i y_j - x_j y_i : \{i,j\} \in E(G)] \subset S.$$

We call $\mathcal{R}(G)$ the *binomial edge ring* of G.

Several kinds of ideals and subrings of polynomial rings associated with graphs have been introduced and studied.

- We call the ideal $(x_i x_j : \{i,j\} \in E(G)) \subset \Bbbk[x_1, \dots, x_d]$ the edge ideal of G. For an introduction to edge ideals and some fundamental results on them, see, e.g., [1, Section 11] and [6, Section 7].
- We call the subring $\Bbbk[x_i x_j : \{i,j\} \in E(G)] \subset \Bbbk[x_1, \dots, x_d]$ the edge ring of G. See, e.g., [2, Section 5] and [6, Sections 10 and 11] for their introduction.
- We call the ideal $(x_i y_j - x_j y_i : \{i,j\} \in E(G)) \subset S$ the binomial edge ideal of G. See, e.g., [2, Section 7] for the introduction.

Considering these trends, it is quite natural to introduce binomial edge rings of G.

2. SAGBI basis and Hibi rings

Fix a monomial order $<$ on the polynomial ring R. Given $\mathcal{F} = \{f_1, \dots, f_n\} \subset R$, consider the finitely generated subalgebra $\Bbbk[\mathcal{F}]$ of R. We say that $\mathcal{F}$ is a *SAGBI basis* with respect to $<$ if $\mathrm{in}_<(\Bbbk[\mathcal{F}]) = \Bbbk[\mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_n)]$ holds, where $\mathrm{in}_<(f)$ denotes the initial monomial of f with respect to $<$ and $\mathrm{in}_<(\Bbbk[\mathcal{F}])$ is the subalgebra generated by $\{\mathrm{in}_<(f) : f \in \Bbbk[\mathcal{F}]\}$. The terminology "SAGBI" derives from "Subalgebra Analogue to Gröbner Basis for Ideals" and was introduced in [4]. It is not necessary that $\mathrm{in}_<(\Bbbk[\mathcal{F}])$ is finitely generated even if $\Bbbk[\mathcal{F}]$ is finitely generated ([4]).

Let $\Pi = \{p_1, \dots, p_{d-1}\}$ be a poset equipped with a partial order $\prec$. A *poset ideal* of Π is a subset $I \subset \Pi$ satisfying "$x \in I$ and $y \prec x$ imply $y \in I$". Let $\mathcal{I}(\Pi)$ denote the set of all poset ideals of Π. We define the $\Bbbk$-subalgebra $\Bbbk[\Pi]$ by setting $\Bbbk[\Pi] := \Bbbk[(\prod_{p_i \in I} x_i) x_d : I \in \mathcal{I}(\Pi)] \subset \Bbbk[x_1, \dots, x_d]$. We call $\Bbbk[\Pi]$ the *Hibi ring* of Π.

3. Initial algebras of Plücker algebras

Given $k, d \in \mathbb{Z}$ with $1 \le k < d$, let $\mathbf{I}_{k,d} = \{I \subset [d] : |I| = k\}$. We define

$$\mathcal{A}_{k,d} := \Bbbk[\det(X_I) : I \in \mathbf{I}_{k,d}] \subset \Bbbk[x_{ij} : 1 \le i \le k, 1 \le j \le d],$$

where X_I denotes the $k \times k$-submatrix of the $k \times d$-matrix of indeterminates (x_{ij}) whose columns are indexed by I. This algebra $\mathcal{A}_{k,d}$ is called *Plücker algebras*, known

Graduate School of Information Science and Technology, Osaka University, Osaka, Japan
higashitani@ist.osaka-u.ac.jp.

as a homogeneous coordinate ring of Grassmannians $\mathrm{Gr}_{\Bbbk}(k,d)$. Moreover, it is known that $\{\det(X_I) : I \in \mathbf{I}_{k,d}\}$ forms a SAGBI basis of $\mathcal{A}_{k,d}$ with respect to the graded lexicographic order induced by the ordering of variables $x_{11} > \cdots > x_{1d} > x_{21} > \cdots > x_{2d} > \cdots > x_{k1} > \cdots > x_{kd}$. Furthermore, the initial algbra $\mathrm{in}_<(\mathcal{A}_{k,d})$ with respect to this monomial order is isomorphic to the Hibi ring of a certain poset. For more detailed information, see, e.g., [5, Chapter 11]. Via initial algebras, we can obtain the Hilbert series, the Gorensteinness of $\mathcal{A}_{k,d}$, and so on.

Here, notice that $\mathcal{A}_{2,d}$ coincides with $\mathcal{R}(K_d)$, where K_d is the complete graph with d vertices. Namely, we can interpret $\mathcal{R}(G)$ as a certain analogue of Plücker algebras. In this talk, we discuss the properties of $\mathcal{R}(G)$ by studying the initial algebras.

4. Main Results

Let $K_{a,b}$ be the complete bipartite graph. The goal of this talk is to determine a SAGBI basis of $\mathcal{R}(K_{a,b})$. To this end, we introduce some notation:

$$S_{a,b} := \Bbbk[x_1,\dots,x_a,x'_1,\dots,x'_b,y'_1,\dots,y'_a,y_1,\dots,y_b]$$

$$f_{ij} := x_iy_j - x'_jy'_i \in S_{a,b} \ \ (1 \le i \le a, 1 \le j \le b)$$

$<$: graded lexicographic order induced by

$$x_1 > \cdots > x_a > x'_1 > \cdots > x'_b > y'_1 > \cdots > y'_a > y_1 > \cdots > y_b$$

Let $\Pi_{a,b}$ be the poset depicted in Figure 1.

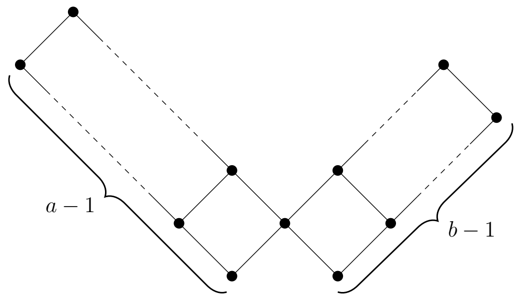

Figure 1. Poset $\Pi_{a,b}$

Theorem 1. *Given $2 \le a \le b$, let*

$$\mathcal{G}_{a,b} = \{f_{ij} : 1 \le i \le a,\ 1 \le j \le b\} \cup \{f_{ij'}f_{i'j} - f_{ij}f_{i'j'} : 1 \le i < i' \le a,\ 1 \le j' < j \le b\}.$$

Then $\mathcal{G}_{a,b}$ forms a SAGBI basis of $\mathcal{R}(K_{a,b})$ with respect to $<$ and the initial algebra of $\Pi_{a,b}$ becomes isomorphic to the Hibi ring of $\Pi_{a,b}$. Hence, we see the following:

- *$\mathcal{R}(K_{a,b})$ is a Cohen–Macaulay domain of Krull dimension $2(a+b-2)$;*
- *$\mathcal{R}(K_{a,b})$ is Gorenstein if and only if $a = 2$ or $a = b$.*

References

[1] J. Herzog and T. Hibi, "Monomial ideals", GTM, **260**, London: Springer, 2011.
[2] J. Herzog, T. Hibi and H. Ohsugi, "Binomial ideals", GTM, **279**, Springer, Cham, 2018.
[3] A. Higashitani, Binomial edge rings of complete bipartite graphs, in preparation.
[4] L. Robbiano and M. Sweedler, Subalgebra bases, Commutative algebra (Salvador, 1988), Lecture Notes in Math., vol. 1430, Springer, Berlin, (1990), pp. 61–87.
[5] B. Sturmfels, Gröbner bases and convex polytopes, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.
[6] R. H. Villarreal, "Monomial algebras, 2nd ed. ed.", Monogr. Res. Notes Math., Boca Raton, FL: CRC Press, 2015.