# Normalized depth and regularity of squarefree powers

Nursel Erey (Gebze Technical University)

**Abstract:** The $k$th squarefree power of a monomial ideal $I$ is generated by the squarefree monomials in $I^k$. If $I$ is an edge ideal, then the $k$th squarefree power of $I$ is closely related to the $k$-matchings of the associated graph. In this talk, we are interested in regularity and normalized depth function of squarefree powers of edge ideals. We discuss the question of when such powers have linear resolutions or are linearly related. We bound the regularity via refined versions of matching numbers and explore nonincreasing behaviour of the normalized depth function. This talk is based on joint work with Jürgen Herzog, Takayuki Hibi and Sara Saeedi Madani in [1, 2, 3].

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# LOCAL COHOMOLOGY MODULES AT IDEALS ASSOCIATED WITH SIMPLICIAL POSETS

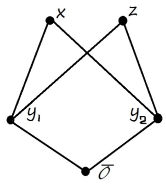
KOHJI YANAGAWA (KANSAI UNIVERSITY)

**This is a joint work with Kosuke Shibata (National Institute of Technology, Yonago College).**

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring, and $I_\Delta \subset S$ a Stanley-Reisner ideal. In [1, 3], we studied the local cohomology modules $H^i_{I_\Delta}(S)$ very precisely. We try to extend this approach to the defining ideal $I_P$ of the face ring of a simplicial poset. ($I_P$ is a generalization of $I_\Delta$.)

We say a finite poset $P$ is *simplicial*, if $P$ has the smallest element $\hat{0}$, and the subposet $\{\, y \in P \mid y \le x \,\}$ is isomorphic to a boolean algebra (i.e., the power set $2^{[n]}$ for some $n$) for each $x \in P$. If we regard (the set of faces of) a simplicial complex as a poset by inclusion, it is simplicial. In general, a simplicial poset is the set of cells of a regular CW complex of certain type. The next example is a simplicial poset which is not a simplicial complex. The corresponding CW complex

$P_1$




is a circle consisting of two segments ($x$ and $z$) and two points ($y_1$ and $y_2$).

Stanley [2] assigned the commutative ring $A_P$ to a simplicial poset $P$. For the construction, we need preparation. First, for $x_1, \dots, x_m \in P$, $[x_1 \vee \cdots \vee x_m]$ denotes the set of minimal elements of common upper bounds of $x_1, \dots, x_m$. If $[x \vee y] \neq \emptyset$ then $\{\, z \in P \mid z \le x, y \,\}$ has the largest element $x \wedge y$. (For the above $P_1$, $[y_1 \vee y_2] = \{x, z\}$ and $y_1 \wedge y_2 = \hat{0}$. But $[x, z] = \emptyset$, and $x \wedge z$ does not exist.) Set $P^* := P \backslash \{\hat{0}\}$, and $S := K[\, t_x \mid x \in P^* \,]$ the polynomial ring over a field $K$. Set

$$f_{x,y} := t_x t_y - t_{x \wedge y} \sum_{z \in [x \vee y]} t_z$$

for $x, y \in P^*$. Here we set $t_{\hat{0}} = 1$ for the convenience, and if $[x \vee y] = \emptyset$, we have $f_{x,y} = t_x t_y$. Note that if $x$ and $y$ are comparable, then $f_{x,y} = 0$. For the ideal

$$I_P := (\, f_{x,y} |\; x, y \in P \,) \subset S,$$

set $A_P := S/I_P$. Let $y_1, \dots, y_n$ be the rank 1 elements of $P$, and set $t_i := t_{y_i}$ for simplicity. Then $I_P$ and $A_P$ admit $\mathbb{Z}^n$-grading with $\deg y_i = \mathbf{e}_i \in \mathbb{N}^n$ for each $i$ ($\mathbf{e}_i$ is the $i$-th coordinate vector). We regard $S$ and $A_P$ are $\mathbb{Z}^n$-graded rings in this way.

For our running example $P_1$, writing $t_x, t_z, \dots$ simply as $x, z, \dots$, we have

$$A_{P_1} = \frac{K[y_1, y_2, x, z]}{(y_1 y_2 - x - z, xz)}.$$

This is a graded ring with $\deg y_1 = (1,0)$, $\deg y_2 = (0,1)$ and $\deg x = \deg z = (1,1)$.

Any $\mathbb{Z}^n$-graded prime ideal of $S$ containing $I_P$ is of the form

$$\mathfrak{p}_x := (t_z \mid z \not\leq x) + I_P$$

for some $x \in P$. Here $\mathfrak{p}_{\hat{0}}$ is the graded maximal ideal $(t_z \mid z \in P^*)$. The quotient ring $S/\mathfrak{p}_x$ is isomorphic to the polynomial ring $S_x := K[t_i \mid y_i \leq x]$. Let $\pi_x : S \to S_x$ be the natural surjection. For the above $P_1$, we have $S_x = K[y_1, y_2]$, $\pi_x(x) = y_1 y_2$ and $\pi_x(z) = 0$.

**Theorem 1.** *The injective envelope of $S_x$ in the category of $\mathbb{Z}^n$-graded $S$-modules is given in the following way: For the simplicity, assume that $x \in [y_1 \vee y_2 \vee \cdots \vee y_m]$, and set $P^*_{-x} := P^* \setminus \{y_1, \ldots, y_m\}$. As $\mathbb{Z}^n$-graded vector spaces, we have*

$$E_x := K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \otimes_K K[t_z^{-1} \mid z \in P^*_{-x}].$$

*Regarding $K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ as an $S$-module through $\pi_x : S \to S_x$, $E_x$ is an $S \otimes_K S$-module. Finally, through the ring homomorphism $\Delta : S \to S \otimes_K S$ defined by*

$$\Delta(t_z) = t_z \otimes 1 + 1 \otimes t_z \qquad (\forall z \in P^*)$$

*for all $z \in P^*$, we regard $E_x$ as an $S$-module.*

**Example 2.** For our running example $P_1$, we have

$$E(S/\mathfrak{p}_x) = K[y_1^{\pm 1}, y_2^{\pm 1}] \otimes_K K[x^{-1}, z^{-1}],$$

and its module structure is given by

$$x \cdot (y_1^a y_2^b \otimes x^{-c} z^{-d}) = y_1^{a+1} y_2^{b+1} \otimes x^{-c} z^{-d} + y_1^a y_2^b \otimes x^{-c+1} z^{-d} \quad (\text{if } c = 0, \text{ then } x^{-c+1} = 0),$$

$$z \cdot (y_1^a y_2^b \otimes x^{-c} z^{-d}) = y_1^a y_2^b \otimes x^{-c} z^{-d+1} \qquad (\text{if } d = 0, \text{ then } z^{-d+1} = 0),$$

$$y_1 \cdot (y_1^a y_2^b \otimes x^{-c} z^{-d}) = y_1^{a+1} y_2^b \otimes x^{-c} z^{-d}, \quad y_2 \cdot (y_1^a y_2^b \otimes x^{-c} z^{-d}) = y_1^a y_2^{b+1} \otimes x^{-c} z^{-d}.$$

If $\psi : E_x \to E_{x'}$ is $\mathbb{Z}^n$-graded, then $\psi(1 \otimes 1) = c \cdot (1 \otimes 1) \in E_{x'}$ for $\exists c \in K$, and hence $\psi(S_x) \subset S_{x'}$. Contrary to the Stanley–Reisner case, even if $\psi(1 \otimes 1) = 1 \otimes 1$, $\psi$ is NOT uniquely determined.

**Theorem 3.** *Let $\varphi : E_x \to E_{x'}$ be a $\mathbb{Z}^n$-graded $S$-homomorphism with $\varphi(1 \otimes 1) = 1 \otimes 1$. After suitable base change of $E_x$, we have*

$$\varphi : E_x \ni 1 \otimes u \longmapsto 1 \otimes u \in E_{x'}$$

*for all $u \in K[t_z^{-1} \mid z \in P^*_{-x}]$ (since $P^*_{-x} \subset P^*_{-x'}$, $u$ also belongs to $K[t_z^{-1} \mid z \in P^*_{-x'}]$).*

Using this result, for a $\mathbb{Z}^n$-graded dualizing complex ${}^*D_S^\bullet$, we can describe $\Gamma_{I_P}({}^*D_S^\bullet)$ explicitly. In this way, we hope argument in [1, 3] for $H^i_{I_\Delta}(S)$ also work for $H^i_{I_P}(S)$, but several problems still remain.

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*Email address*: yanagawa@kansai-u.ac.jp

# $h$-function of local rings of characteristic $p$

Cheng Meng

**Abstract:** For a Noetherian local ring $R$ of characteristic $p$, there are two important numerical invariants: Hilbert-Kunz multiplicity and the F-signature. They are asymptotic colengths of certain sequences of ideals and they quantify the severity of singularities at a point of a variety. Actually, the Hilbert-Kunz multiplicity of a ring is always a real number at least 1 while the F-signature of a ring is always a nonnegative real number at most 1. Whether they are exactly equal to 1 characterizes regularity under mild assumptions. They also give characterizations of different objects in commutative algebra in characteristic $p$ like the tight closure and the $F$-regularity.

Although they are important in prime characteristic commutative algebra, the concrete computation of the Hilbert-Kunz multiplicity and the F-signature is very difficult. In the graded case, the theory of Hilbert-Kunz multiplicity has witnessed two new generalizations in recent years: the Hilbert-Kunz density function and the Frobenius-Poincaré function. The main idea behind these generalizations is that instead of computing the total colengths of ideals, we can keep track of the colengths in each degree. As the degree grows linearly with the indices of the ideals, the colengths in these degrees possess certain asymptotic behavior which is similar to the Hilbert-Kunz multiplicity. The study of the Hilbert-Kunz density function and the Frobenius-Poincaré function introduces methods in algebraic geometry since taking Proj of a standard graded ring gives a projective variety. Therefore, we can get a finer description of the Hilbert Kunz multiplicity.

One question arises naturally in the study of the Hilbert-Kunz density function in graded case: can we find a counterpart in the local setting? In this talk, we will introduce a numerical invariant called $h$-function that answers this question. It is a function of a real variable $s$ that estimates the asymptotic behavior of the sum of an ordinary power and a Frobenius power. We will point out that the derivative of $h$-function, which exists almost everywhere, is the local analogue of the Hilbert-Kunz density function.

After introducing the $h$-function, we will mention the following properties of $h$-function or functions related to $h$-function:

(1) Existence in a very general setting;
(2) Good analytic property, including certain convexity property and almost everywhere second differentiability;
(3) Recovering the Hilbert-Kunz multiplicity, the $F$-signature and the $F$-threshold;
(4) Multiplicity-like additivity on exact sequences;
(5) Multiplicativity on Segre products.

There are also other efforts to develop multiplicity-like numerical invariants in characteristic $p$. One particular invariant is called the $s$-multiplicity which is a mixture of the Hilbert-Samuel and the Hilbert-Kunz multiplicity. We will show that both the Hilbert-Kunz density function and the $s$-multiplicity are equivalent to the $h$-function, therefore the computational result of one of them will give the results of the other two immediately.

# DUAL $F$-SIGNATURES OF VERONESE SUBRINGS AND SEGRE PRODUCTS OF POLYNOMIAL RINGS

KOJI MATSUSHITA

This talk is based on [6]. Let $(R,\ \mathfrak{m})$ be a reduced Noetherian local ring of prime characteristic $p > 0$ and assume that the residue field $R/\mathfrak{m}$ is algebraically closed. In this situation, we can define the $e$-times iterated Frobenius morphism $F^e : R \to R$ $(r \mapsto r^{p^e})$ for $e \in \mathbb{Z}_{>0}$. For an $R$-module $M$, let $F^e_* M$ denote the *Frobenius push-forward* of $M$, which is an $R$-module given by restriction of scalars under $F^e$ (that is, $F^e_* M$ is just $M$ as an abelian group and its $R$-module structure is defined by $r \cdot m := F^e(r)m = r^{p^e}m \ \ (r \in R,\ m \in M)$). We say $R$ is $F$*-finite* if $F^e_* R$ is a finitely generated $R$-module. In this talk, we always assume that $R$ is $F$-finite and has the canonical module $\omega_R$ since we only discuss such rings.

In [9], Sannai introduced the following value for an $R$-module $M$:

$$s(M) := \limsup_{e\to\infty} \frac{\max\{N : \text{ there is a surjection } F^e_* M \twoheadrightarrow M^{\oplus N}\}}{\operatorname{rank} F^e_* M}. \tag{0.1}$$

The value $s(R)$ is called the $F$*-signature* of $R$ ([5, 12]). In addition, we call $s(\omega_R)$ the *dual* $F$*-signature of* $R$ ([9, 11]) and write it by $s_{\mathrm{dual}}(R)$.

Regarding $s(R)$ and $s_{\mathrm{dual}}(R)$, the following facts are known:

**Theorem 0.1** (see [1, 5, 9, 11, 13])**.** *Let $(R, \mathfrak{m})$ be a reduced $F$-finite Cohen–Macaulay local ring with* $\operatorname{char} R = p > 0$ *and assume that $R/\mathfrak{m}$ is an algebraically closed field. Then we have the following.*

(1) $0 \le s(R) \le s_{\mathrm{dual}}(R) \le 1$.
(2) *The following are equivalent:*
  (a) *$R$ is regular;*
  (b) *$s(R) = 1$;*
  (c) *$s_{\mathrm{dual}}(R) = 1$.*
(3) *$R$ is strongly $F$-regular if and only if $s(R) > 0$.*
(4) *$R$ is $F$-rational if and only if $s_{\mathrm{dual}}(R) > 0$.*
(5) *$R$ is Gorenstein if and only if $s(R) = s_{\mathrm{dual}}(R)$.*

This theorem shows that $s(R)$ and $s_{\mathrm{dual}}(R)$ measure the severity of singularities of $R$. Therefore, it is important to determine these invariants and consider what their explicit values mean. Indeed, $F$-signatures (resp. dual $F$-signatures) have been given for several classes of commutative rings with positive characteristic, see, e.g., [3, 4, 5, 10, 13] (resp. [7, 8, 9]).

In this talk, we present new examples where the dual $F$-signatures are computed. Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 0$. First, we completely determine the dual $F$-signatures of Veronese subrings:

**Theorem 0.2.** *Let $V_{n,d}$ be the nth Veronese subring of the polynomial ring over $\Bbbk$ with d variables. Then we have*

$$s_{\mathrm{dual}}(V_{n,d}) = \frac{1}{d}\left\lceil \frac{d}{n} \right\rceil.$$

This result was proved in a different way at the same time in [2].

Moreover, we give the dual $F$-signatures of Segre products of two polynomial rings:

**Theorem 0.3.** *Let $S_{r_1,r_2} := \Bbbk[x_{1,1},\dots,x_{1,r_1+1}]\#\Bbbk[x_{2,1},\dots,x_{2,r_2+1}]$ with $r_1 \le r_2$, and let $d = r_1 + r_2 + 1$. Then we have*

$$s_{\mathrm{dual}}(S_{r_1,r_2}) = \frac{\sum_{l=r_1}^{r_2} \binom{l}{r_1} A_{l,d}}{\binom{r_2}{r_1} d!},$$

where $A_{l,d}$ denotes the *Eulerian number*, which is the number of permutations of the numbers 1 to $n$ in which exactly $k$ elements are less than the previous element.

Moreover, we give an upper bound for the dual $F$-signatures of Segre products of three or more polynomial rings by using their generalized $F$-signatures. If time permits, we will introduce it too.

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Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, Japan

*Email address*: k-matsushita@ist.osaka-u.ac.jp

# An upper bound for the Frobenius number and stretched numerical semigroups

Kazufumi Eto, Naoyuki Matsuoka,
Takahiro Numata, Kei-ichi Watanabe

Let $H$ be a *numerical semigroup*, which is a submonoid of $\mathbb{N} = \{0, 1, 2, \ldots\}$ with $|\mathbb{N} \setminus H| < \infty$. The largest integer in $\mathbb{Z} \setminus H$ is called the Frobenius number of $H$ and is denoted by $\mathrm{F}(H)$. Determining $\mathrm{F}(H)$ is known as the *Frobenius Coin Problem.*

Consider $H = \langle n_1, n_2, \ldots, n_e \rangle = \{\lambda_1 n_1 + \cdots + \lambda_e n_e \mid \lambda_1, \ldots, \lambda_e \in \mathbb{N}\}$, a numerical semigroup, and let $k$ be a field. The rings

$$k[[H]] = k[[t^{n_1}, \ldots, t^{n_e}]] \subseteq k[[t]] \text{ and } k[H] = k[t^{n_1}, \ldots, t^{n_e}] \subseteq k[t]$$

are called the *numerical semigroup rings* of $H$ over $k$, where $t$ is an indeterminate over $k$. The Frobenius number $\mathrm{F}(H)$ of $H$ corresponds to the $a$-invariant of the graded ring $k[H]$, making the Frobenius Coin Problem significant in both numerical semigroup theory and commutative algebra.

In what follows, $H = \langle n_1, n_2, \ldots, n_e \rangle$ be a numerical semigroup with $n_1 < n_2 < \cdots < n_e$. In this talk, we provide a new, simple upper bound for $\mathrm{F}(H)$ under a mild assumption as follows:

**Theorem 1.** *Suppose that* $\gcd(n_1, n_i) = 1$ *for some* $2 \leq i \leq e$. *Then* $\mathrm{F}(H) \leq (n_1 - 1)n_i - (e - 1)n_1$.

Additionally, we found that, roughly speaking, the condition that the ring $k[[H]]$ is stretched imposes a significant restriction on the behavior of $\mathrm{F}(H)$. In order to state our results precisely, we define the notion of *stretched numerical semigroups.*

The concept of stretched local rings was introduced by Sally [1]. An Artinian local ring $(A, \mathfrak{m})$ is said to be *stretched* if $\mu_A(\mathfrak{m}^2) \leq 1$, where $\mu_A(X)$

denotes the number of a minimal system of generators of $A$-module $X$. A Cohen-Macaulay local ring $(A, \mathfrak{m})$ is said to be *stretched* if there exists a parameter ideal $Q$ of $A$ such that $Q$ is a reduction of $\mathfrak{m}$ and $A/Q$ is a stretched Artinian local ring.

Now, let us consider a numerical semigroup ring $k[[H]]$. For $f \in \mathfrak{m} = (t^{n_1}, t^{n_2}, \ldots, t^{n_e})$, $(f)$ is a reduction of $\mathfrak{m}$ if and only if $v(f) = n_1$, where $v$ denotes the discrete valuation on $k[[t]]$. Therefore, $k[[H]]$ is a stretched local ring if and only if there exists $f \in \mathfrak{m}$ such that $v(f) = n_1$ and $k[[H]]/(f)$ is stretched. Regarding this property for $k[[H]]$, we present two examples to be noticed.

**Example 2.** (1) Let $H = \langle 7, 11, 24, 26 \rangle$. Then $k[H]/(t^7)$ is not stretched, but $k[[H]]/(t^7 - t^{11})$ is stretched.

(2) Let $H = \langle 7, 11, 26, 30 \rangle$. Then $k[[H]]/(t^{11})$ is stretched. In this case, $k[[H]]/(t^7)$ is not stretched, while $k[[H]]/(t^7 - t^{11})$ is stretched.

It is unclear if the existence of $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $k[[H]]/(f)$ is stretched implies that $k[[H]]$ is stretched in the sense of Sally [1]. Thus, we define stretched numerical semigroups as follows, removing the assumption on the value of $f$:

**Definition 3.** $H$ is said to be *stretched*, if $k[[H]]/(f)$ is a stretched Artinian local ring for some $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

We can then state the additional main theorems of this talk:

**Theorem 4.** *Suppose* $\gcd(n_1, n_2) = 1$ *and* $H$ *is stretched. Then*

$$(n_1 - e + 1)n_2 - n_1 \leq \mathrm{F}(H) \leq (n_1 - 1)n_2 - (e - 1)n_1.$$

**Theorem 5.** *Suppose* $\gcd(n_1, n_2) = 1$. *Then* $\mathrm{F}(H)$ *attains the upper bound in Theorem 1 if and only if* $k[[H]]/(t^{n_2})$ *is stretched.*

We also discuss possible values of $\mathrm{F}(H)$ for fixed $n_1$ and $n_2$.

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# THE V-NUMBERS OF SQUAREFREE MONOMIAL IDEALS

TATSUYA KATAOKA, YUJI MUTA AND NAOKI TERAI (OKAYAMA UNIVERSITY)

In 2020, Cooper, Seceleanu, Toăneanu, Pinto and Villarreal [2] introduced the notion of the v-*number* within the context with coding theory. In the field of combinatorial commutative algebra, the case that the squarefree monomial case is mainly treated and the relation with the Castelnuovo-Mumford regularity has been considered; see [1, 3, 6, 7]. Let us recall the definition of the v-number.

**Definition 1.1.** Let $S = k[x_1, \dots, x_n]$ be a polynomial ring with $\deg x_i = 1$ for all $i \in [n] = \{1, 2, \dots, n\}$ and let $I$ be a homogeneous ideal of $S$. The v-*number* of $I$ is defined to be

$$\mathrm{v}(I) = \min\{\mathrm{v}_P(I) \,:\, P \in \mathrm{Ass} S/I\}$$

where $\mathrm{v}_P(I) := \min\{\deg f \,:\, (I : f) = P\}$.

**Problem 1.2.** *Let $I$ be a squarefree monomial ideal. When does $\mathrm{v}(I) \le \mathrm{reg} S/I$ hold?*

First, we obtained a characterization of the v-number of the Stanley–Reisner ideals of simplicial complexes in terms of their Alexander dual.

**Theorem 1.3.** *Assume $P = (x_1, x_2, \dots, x_h)$ be an associated prime of $I_\Delta$. Then*

$$\mathrm{v}_P(I_\Delta) = n - h - \max\{|\cap_{i=1}^h F_i| \,:\, F_i \in \Delta^\vee \text{ such that } \{x_1, \dots, \hat{x_i}, \dots, x_h\} \subset F_i \text{ for all } i\}.$$

Also, we obtained the v-number with pure height two in terms of graded betti numbers of $\Bbbk[\Delta]$ as a corollary of Theorem 1.3.

**Corollary 1.4.** *Let $\Delta$ be a pure simplicial complex with $\mathrm{ht} I_\Delta = 2$. Then $\mathrm{v}(I_\Delta) = \min\{j \,:\, \beta_{2,2+j}(\Bbbk[\Delta]) \neq 0\}$.*

Moreover, we obtained the v-number of Stanley-Reisner ideals that are 2-Cohen-Macaulay, in particular, Gorenstein, and of a matroid complex, respectively.

**Theorem 1.5.** *Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. Then $\mathrm{v}(I_\Delta) = d$ if and only if $\Delta$ is 2-pure, where $\Delta$ is 2-pure means that $\Delta$ is pure and $\Delta_{V\setminus\{x\}} = \{F \in \Delta \,:\, x \notin F\}$ is pure with $\dim \Delta = \dim \Delta_{V\setminus\{x\}}$ for any $x \in V$. In particular, If $\Delta$ is 2-Cohen–Macaulay, Gorenstein or matroid, then $\mathrm{v}(I_\Delta) = d$.*

For a graph theory, Theorem 1.3 provides a lot of corollaries. Let us recall the structure of very-well covered graphs. Let $H$ be a Cohen–Macaulay very-well covered graph. Then $H(n_1, \dots, n_{d_0})$ to $H$, where $n_1, \dots, n_{d_0}$ are positive integers is defined by $V(H(n_1, \dots, n_{d_0})) = \bigcup_{i=1}^{d_0} \left(\{x_{i1}, \dots, x_{in_i}\} \cup \{y_{i1}, \dots, y_{in_i}\}\right)$ as the vertex set and the edge set is obtained by replacing the edges $x_1y_1, \dots, x_{d_0}y_{d_0}$ in $H$ with the complete bipartite graphs $K_{n_1,n_1}, \ldots, K_{n_{d_0},n_{d_0}}$, respectively. From [4, Theorem 3.5] it is known that for a very well-covered graph $G$ on the vertex set $X_{[d]} \cup Y_{[d]}$, there exist positive integers $n_1, \ldots, n_{d_0}$ with $\sum_{i\in[d_0]} n_i = d$ and a Cohen–Macaulay very well-covered graph $H$ on the vertex set $X_{[d_0]} \cup Y_{[d_0]}$ such that $G \cong H(n_1, \ldots, n_{d_0})$. Also, let us recall the definition of multi-whisker graphs which is introduced in [5]. Let $G_0$ be a graph on the vertex set $X_{[h]} = \{x_1, \ldots, x_h\}$. Then the *multi-whisker graph associated with* $G_0$ is the graph $G = G_0[n_1, \ldots, n_h]$ on the vertex set $V(G) = X_{[h]} \cup Y$, where $Y = \{y_{1,1}, \ldots, y_{1,n_1}\} \cup \cdots \cup \{y_{h,1}, \ldots, y_{h,n_h}\}$ and the edge set $E(G) = E(G_0) \cup \{x_1y_{1,1}, \ldots, x_1y_{1,n_1}, \ldots, x_hy_{h,1}, \ldots, x_hy_{h,n_h}\}$.

**Corollary 1.6.** *For a graph $G$ on the vertex set $V$,*

$$\mathrm{v}(J(G)) = \min\{|V \setminus F \cap F'| - 2 \ : \text{there exist } \{x_i, x_j\} \in E(G) \text{ and } F, F' \in \mathcal{F}(\Delta(G)) \text{ such that } x_i \in F \setminus F' \text{ and } x_j \in F' \setminus F\},$$

*where $\mathcal{F}(\Delta(G)) = \{A \ : A$ is a maximal independent set of $G\}$.*

**Corollary 1.7.** *Let $G = H(n_1, n_2, \ldots, n_{h_0})$ be a very well-covered graph with $\mathrm{ht} I(G) = h$, where $H$ is a Cohen-Macaulay very well-covered graph. Then we have $\mathrm{v}(J(G)) = h + \min\{n_1, n_2, \ldots, n_{h_0}\} - 2$.*

**Corollary 1.8.** *Let $G = G_0[n_1, n_2, \ldots, n_h]$ be the multi-whisker graph associated with $G_0$. Then we have $\mathrm{v}(J(G)) = h + \min\{n_1, n_2, \ldots, n_h\} - 2$.*

Moreover, we obtained the v-number of edge ideals of very-well covered graphs and multi-whisker graphs as a generalization of [3, Theorem 3.20]

**Theorem 1.9.** *Let $G$ be a very well-covered graph. Then we have $\mathrm{v}(I(G)) \leq \mathrm{reg} S/I(G)$.*

**Theorem 1.10.** *Let $G = G_0[n_1, n_2, \ldots, n_h]$ be the multi-whisker graph associated with $G_0$. Then we have $\mathrm{v}(I(G)) = \min\{|A| \ : A$ is a maximal independent set of $G_0\}$.*

In this talk, we will explain the results discussed above and give the answers to [6, Question 3.12] and [7, Question 5.5].

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