MODULE SCHEMES IN INVARIANT THEORY

HOLGER BRENNER

ABSTRACT. Let G be a finite group acting linearly on the polynomial ring with invariant ring R. We assign, to a linear representation of G, a corresponding quotient scheme over Spec R, and we show how to reconstruct the action from the quotient scheme. This works in particular in the case of a reflection group, where Spec R itself is an affine space, in contrast to the Auslander correspondence, where one has to assume that the basic action is small, i.e. contains no pseudo reflection. These quotient schemes exhibit rich geometric features which mirror properties of the representation. In order to understand the image of this construction, we encounter module schemes (a forgotten notion of Grothendieck), module schemes up to modification and fiberflat bundles.

ON AN ALGEBRA WHOSE QUOTIENT FIELD IS RETRACT RATIONAL

TAKANORI NAGAMINE

This talk is based on a joint work with Neena Gupta ([2]). Throughout the paper, all rings are commutative with unity and any domain is understood to be an integral domain. Let R be a domain and A, B be R-algebras. R^* is the group of units of R. $Q(R)$ is the quotient field of R. For an integer $n \geq 0$, $R^{[n]}$ denotes the polynomial ring in n variables over R. For $f \in R$, $R_f = S^{-1}R$ where $S = \{1, f, f^2, \ldots\}$. For a field K, $K^{(n)}$ denotes the field of fractions of the polynomial ring $K^{[n]}$. A is called an **R-retract** of B if there are R-algebra homomorphisms $\varphi : B \to A$ and $\psi : A \to B$ such that $\varphi \circ \psi = \mathrm{id}_A$. In particular, φ is called an **R-retraction**. When A is a domain, $\mathrm{tr.deg}_R A$ denotes the transcendence degree of $Q(A)$ over $Q(R)$.

Let K/k be a field extension. K is called **rational** over k if $K \cong_k k^{(n)}$ for some $n \geq 0$. K is called **stably rational** over k if $K^{(m)}$ is rational for some $m \geq 0$. K is called **retract rational** over k if there exists a k-domain A such that $Q(A) = K$ and A is a k-retract of $k[x_1, \ldots, x_n]_f$ for some $n \geq 0$ and $f \in k[x_1, \ldots, x_n] \cong_k k^{[n]}$. It is well known that if k is an infinite field, then "rational" $\implies$ "stably rational" $\implies$ "retract rational" (see e.g., [3, Proposition 3.6 (a)]). The following question is an analogue of [1, Question 4].

Question 1.1. Let $M \in k[x_1, \ldots, x_n] \cong_k k^{[n]}$ be a monomial in $x_1, \ldots, x_n$ and A be a k-retract of $k[x_1, \ldots, x_n]_M$. Does it follow that $Q(A)$ is rational over k?

Note that, if A is a k-retract of $k[x_1, \ldots, x_n]_M$, then $Q(A)$ is retract rational over k. We give a partial answer of Question 1.1 as below.

Theorem 1.2. *Let k be a field, $M \in k[x_1, \ldots, x_n] \cong_k k^{[n]}$ be a monomial in $x_1, \ldots, x_n$ and A be a k-retract of $B := k[x_1, \ldots, x_n]_M$. If one of the following holds, then $Q(A)$ is rational over k.*

- **(a)** $\mathrm{tr.deg}_k A \in \{0, 1, n\}$.
- **(b)** $\mathrm{rank}(B^*/k^*) \geq n - 2$.
- **(c)** $n \leq 3$.

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(Takanori Nagamine) Department of Mathematics, College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

Email address: nagamine.takanori@nihon-u.ac.jp

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Golod rings and syzygies of the residue fields

Toshinori Kobayashi *

This talk is based on joint work with D. T. Cuong, H. Dao, D. Eisenbud, C. Polini, and B. Ulrich [2]. Throughout, $(R, \mathfrak{m}, k)$ denotes a commutative Noetherian local ring. It is well-known that there is a coefficient-wise inequality:

$$\sum_{n \geq 0} \operatorname{Tor}_n^R(k, k) t^n \leq \frac{(1+t)^e}{1 - \sum_{i=1}^{e - \operatorname{depth} R} \dim_k \mathrm{H}_i(K^R) t^{i+1}}, \tag{0.1}$$

where e is the embedding dimension, and $\mathrm{H}_i(K^R)$ is the i-th Koszul homology on $\mathfrak{m}$. Recall that R is called *Golod* if equality holds in (0.1). Refer to [1] for examples and basic properties of Golod rings. Our goal is to investigate this class of rings further and to uncover new insights into their structure and behavior. For an integer $n \geq 0$, let $\Omega_R^n k$ be the n-th syzygy of the residue field k. In the talk, we will explain our result on the structure of the R-module $\Omega_R^n k$ when R is a Golod ring. Our main theorem reveals that $\Omega_R^{e+1} k$ decomposes as a direct sum of lower syzygies of k. As an application, in the case of codimension 2, we obtain the complete description of all indecomposable direct summands of $\Omega_R^n k$ for any n.

References

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*School of science and technology, Meiji University, tkobayashi@meiji.ac.jp