HOMOLOGICAL PROPERTIES OF F-MODULES

WENLIANG ZHANG

ABSTRACT. The theory of F-modules was introduced by Lyubeznik in 1997 and was built on previous work by Peskine–Szpiro, Hartshorne–Speiser and Huneke–Sharp. Since its introduction, it has found numerous applications in the study of commutative algebra and algebraic geometry in prime characteristic p. In this talk, we will review some background and survey some recent developments on the homological properties of F-modules.

Holonomic D-modules and Noetherian differential operators
for local cohomology classes associated to primary ideals

S. Tajima (Niigata Univ. Japan)

In the 1930s, W. Gröbner addressed the problem of characterizing ideal membership with differential operators. Later, in 1961, L. Ehrenpreis announced "A fundamental principle for systems of linear differential equations with constant coefficients and some of its appications". and gave in particular a description of primary ideals and modules in terms of differential operators. These operators are, nowadays, called Noetherian differential operators.

Noetherian differential operatos have been studied by several authors. V. P. Palamodov, J. -E. Björk, L. Hörmander, O. Liess, U. Oberst, I. Nonkań. We mention only a few.

In 2007, A. Damiano, I. Sabadini and D. C. Struppa presented an algorithm for computing Noetherian differential operators. More recently Y. Cid-Ruiz investigates Noetherian differential operators and implements, with his collaborators, algorithms for computing Noetherian differential operatos.

In this talk, we introduce the concept of Noetherian differential operators for local cohomology classes associated to a primary ideal. We show that formal adjoints of Noetherian differential operators for local cohomology classes give rise to the classical Noetherina differential operators in the sense of Ehrenpreis-Palamodov.

We first recall some basics on Grothendieck local duality on residues. Then, we consider local cohomology classes associated to zero-dimensional primary ideals of polynomial ring. By using the theory of holonomic D-modules, we introduce the concept of Noetherian differential operators for local cohomology classes. By applying Grothendieck local duality, we show that the formal adjoints of these differential operators give rise to the classical Noetherian differential operators. Lastly, we address the positive dimensional case. The key of our approach is the use of the theory of D-modules.

UPPER BOUNDS OF NORMAL REDUCTION NUMBERS OF INTEGRALLY CLOSED IDEALS IN A 2-DIMENSIONAL NORMAL RINGS.

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This talked is based on our joint work in progress ([3]).

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal of A.
In general, if $J \subset I$ are ideals, we say I is **integral** over J if

$$I^{r+1} = JI^r$$

for some r. We say I is **integrally closed** if $I \subset I'$ and I' is integral over I, then $I' = I$.
If I is $\mathfrak{m}$-primary and $A/\mathfrak{m}$ is an infinite field, then there is $Q \subset I$, Q is a parameter ideal and I is integral over Q. We call such Q a **minimal reduction** of I.
For an integrally closed $\mathfrak{m}$-primary ideal I, the **normal reduction numbers** $\mathrm{nr}(I)$ and $\bar{\mathrm{r}}(I)$ are defined by

$$\mathrm{nr}(I) = \min\{n \mid \overline{I^{n+1}} = Q\overline{I^n}\}, \quad \bar{r}(I) = \min\{n \mid \overline{I^{N+1}} = Q\overline{I^N}, \forall N \geq n\}$$

and

$$\mathrm{nr}(A) = \max_{I \subset \mathfrak{m}}\{\mathrm{nr}(I)\}, \quad \bar{r}(A) = \max_{I \subset \mathfrak{m}}\{\bar{r}(I)\}.$$

Needless to say that these invariants of A have very strong effect to determine the structure of the ring A.

In this talk, let A be an excellent normal 2-dimensional local ring and I be an integrally closed $\mathfrak{m}$-primary ideal. We assume k is algebraically closed and $k \subset A$.

We want to analyze properties of normal reduction numbers using resolution of singularities.

For any resolution of singularity $f : X \to \mathrm{Spec}(A)$, we denote

$$\mathbb{E} = f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{r} E_i$$

the exceptional curve of f. We say $Z \in \bigoplus_{i=1}^{r} \mathbb{Z}E_i$ a cycle and for Z, we define

$$p_a(Z) = \frac{Z^2 + K_X Z}{2} + 1$$

We define

$$p_g(A) = \dim_k H^1(X, \mathcal{O}_X) \quad \text{and} \quad p_a(A) = \max_{Z \geq 0} p_a(Z)$$

and call **geometric genus** and **arithmetic genus** of A. We have always $p_a(A) \leq p_g(A)$. Note that $p_a(A)$ is a **topological invariant** of $(X, \mathbb{E})$.

Definition. (1) A is a **rational singularity** iff $p_g(A) = 0$, iff $p_a(A) = 0$.
(2) A is an **elliptic singularity** iff $p_a(A) = 1$. ($p_g(A)$ of an elliptic singularity can be arbitrary large.)
We have showed in our previous works
Theorem. (1) ([2]) A is a rational singularity if and only if $\bar{\mathrm{r}}(A) = 1$.
(2) ([1]) If A is an elliptic singularity, then $\bar{\mathrm{r}}(A) = 2$.

If I is an $\mathfrak{m}$-primary ideal of A, we can take an resolution $f : X \to \mathrm{Spec}(A)$, so that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ and $I = H^0(X, \mathcal{O}_X(-Z))$. In this case, we denote

$$I = I_Z.$$

The cohomology group $H^1(X, \mathcal{O}_X(-Z))$ is very important in our theory and we denote

$$q(I) = \dim_k H^1(X, \mathcal{O}_X(-Z)) \text{ and } q(nI) = \dim_k H^1(X, \mathcal{O}_X(-nZ)).$$

We can show that

$$\dim_k(\overline{I^{n+1}}/Q\overline{I^n}) = q((n+1)I) - 2q(nI) + q((n-1)I).$$

The main result of this talk is;

Main Theorem. $\bar{r}(A) \leq p_a(A) + 1$.

I will talk about also other topics concerning these normal reduction numbers of A.

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[2] Tomohiro Okuma, Kei-ichi Watanabe, and Ken-ichi Yoshida, *Good ideals and p_g-ideals in two-dimensional normal singularities*, Manuscripta Math. **150** (2016), no. 3-4, 499–520.

[3] Improvement of Röhr's vanishing theorem and the normal reduction number for normal surface singularities, in preparation.

NORMAL COHEN-MACAULAY REES ALGEBRA VIA MULTIPLIER/TEST IDEALS

KENTA SATO

This is based on joint work with Hailong Dao, Ilya Smirnov and Shunsuke Takagi.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay reduced local ring of dimension $d \geqslant 1$ essentially of finite type over a field k and I be an $\mathfrak{m}$-primary integrally closed ideal. The main object of this talk is the *Rees algebra*

$$\mathcal{R}(I) := \bigoplus_{n \geqslant 0} I^n t^n \subseteq R[t].$$

The singularities of this algebra has been long studied for its significance in both commutative ring theory and algebraic geometry. In characteristic zero, the following property is essentially proved by Hyry.

Proposition 1 (Hyry)**.** *If* $\mathrm{ch}(k) = 0$*, then the following conditions are equivalent:*

(1) $\mathcal{R}(I)$ *has only rational singularities and the* a-invariant $a(\mathrm{Gr}_I(A))$ *of the tangent cone*

$$\mathrm{Gr}_I(A) := \bigoplus_{n \geqslant 0} I^n / I^{n+1}$$

satisfies $a(\mathrm{Gr}_I(A)) \leqslant 1 - d$.

(2) *The* multiplier submodule $\mathcal{J}(\omega_A, I^{d-2}) \subseteq \omega_A$ *is equal to* ω_A.

A positive characteristic analogue of this proposition was studied by Hara, Watanabe and Yoshida. In this talk, we consider the following condition

$$(1') : \mathcal{R}(I) \text{is normal Cohen-Macaulay, and } a(\mathrm{Gr}_I(R)) \leqslant 1 - d$$

which is weaker than (1), and we give equivalent conditions for (1').

Main Theorem. *If* $\mathrm{ch}(k) = 0$*, then the following conditions are equivalent:*

(1') $\mathcal{R}(I)$ *is normal Cohen-Macaulay, and* $a(\mathrm{Gr}_I(A)) \leqslant 1 - d$.

(2') $\mathcal{J}(\omega_A, I^{d-1}) + (x_1, \dots, x_{d-1})\omega_A \supseteq I\omega_A$ *for some* $x_1, \dots, x_{d-1} \in I$.

(3') *For every (or equivalently, some) minimal reduction* $J \subseteq I$*, we have* $\overline{r}_J(I) \leqslant 1$*, where* $\overline{r}_J(I)$ *is the normal reduction number*

$$\overline{r}_J(I) := \max\{r \geqslant 0 \mid \forall n \geqslant r,\ \overline{I}^{n+1} = J\overline{I}^n\}.$$

We also give several related results. For example, we give a positive characteristic analogue of this result.

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UNIFORM POSITIVITY OF F-SIGNATURE OF MODULO p REDUCTION

TATSUKI YAMAGUCHI

This talk is based on joint work with Shunsuke Takagi.

The F-signature is an invariant of Noetherian F-finite rings of positive characteristic introduced by Huneke and Leuschke [HL02]. By Kunz's theorem, a ring R of positive characteristic is regular if and only if the Frobenius morphism $F : R \to R$ is flat. The F-signature $s(R)$ of a ring R measures how close the Frobenius map F is to being flat. In particular, it is known that R is regular if and only if $s(R) = 1$, and R is strongly F-regular if and only if $s(R) > 0$.

Strongly F-regular singularity, defined in terms of the Frobenius morphism, is a positive characteristic analogue of Kawamata log terminal (klt) singularities, which plays an important role in birational geometry. A normal complex variety X is said to have *strongly F-regular type* if its reduction modulo $p > 0$ is strongly F-regular for $p \gg 0$. Hara and Watanabe [HW02] proved that if the variety X is $\mathbb{Q}$-Gorenstein, then X is of strongly F-regular type if and only if X has log terminal singularities.

The aim of this talk is to study F-signature of reductions modulo $p > 0$. A normal local domain R essentially of finite type over $\mathbb{C}$ is said to have *klt type* if there exists an effective $\mathbb{Q}$-divisor Δ on $\operatorname{Spec} R$ such that $(\operatorname{Spec} R, \Delta)$ is a klt pair. Then R_p, the reduction modulo $p \gg 0$ of R, is strongly F-regular as mentioned above. Hence, we obtain $s(R_p) > 0$ for $p \gg 0$. The following is our main problem.

Question 1 ([CRST18, Question 5.9]). Does there exist a constant $C > 0$ such that $s(R_p) \geqslant C$ for $p \gg 0$?

It is also conjectured that $\lim_{p\to\infty} s(R_p)$ exists. We expect that the limit F-signature $\lim_{p\to\infty} s(R_p)$ detects singularities of klt type. To ignore the convergence problem, we define the F-signature of R as follows.

Definition 2. For R as above, $s(R) = \liminf_{p\to\infty} s(R_p)$.

Note that $s(R) > 0$ is equivalent to the condition in Question 1.

A ring homomorphism $R \to S$ is said to be *pure* if for any R-module M, the canonical morphism $M \to M \otimes S$ is injective. For example, if R is a direct summand of S as an R-module, then the inclusion $R \to S$ is pure. If R and S are essentially of finite type over $\mathbb{C}$, and S is regular, then R is of klt type ([Zhu24]).

Our main theorem is stated as follows:

Theorem 3. *Let $R \to S$ to be a pure local $\mathbb{C}$-algebra homomorphism. Suppose that S is of klt type and $s(S) > 0$. Then $s(R) > 0$.*

Corollary 4. *If there exists a pure local $\mathbb{C}$-algebra homomorphism $R \to S$ with S regular, then $s(R) > 0$.*

A key tool to the proof of the main theorem is α_F-invariant introduced by [Pan23], which is an F-singularity analogue of α-invariant, an invariant related to K-stability. Although Pande defined α_F-invariant for graded rings and Fano varieties, we extend the definition of α_F-invariant to local rings. There exists an inequality concerning the α_F-invariant and the F-signature. When comparing the α_F-invariants of R and S, we need a more complicated technique than reduction modulo $p > 0$, namely, ultraproducts.

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Indecomposability of graded modules over a graded ring
Mitsuyasu Hashimoto (Osaka Metropolitan University)

This is a joint work with Yuntian Yang `arXiv:2306.1423v1` (to appear in J. Math. Kyoto Univ). The purpose of this talk is to give a proof to the following theorem.

Theorem 1. *Let $R = \bigoplus_{i\geq 0} R_i$ be a Noetherian commutative non-negatively graded ring such that $(R_0, \mathfrak{m}_0)$ is a Henselian local ring. Let $\mathfrak{m}$ be its unique graded maximal ideal $\mathfrak{m}_0 + \bigoplus_{i>0} R_i$. Let T be a module-finite (non-commutative) graded R-algebra. Let $T\,\mathrm{grmod}$ denote the category of finite graded left T-modules, and $M \in T\,\mathrm{grmod}$. Then the following are equivalent:*

(1) *$\hat{M}$ is an indecomposable $\hat{T}$-module, where $\widehat{(-)}$ denotes the $\mathfrak{m}$-adic completion;*

(2) *$M_\mathfrak{m}$ is an indecomposable $T_\mathfrak{m}$-module;*

(3) *M is an indecomposable T-module;*

(4) *M is indecomposable as a graded T-module.*

As a corollary, we prove the following.

Corollary 2. *Let $R = \bigoplus_{i\geq 0} R_i$ and T be as in the theorem. For two finite graded left T-modules M and N, the following are equivalent:*

(1) *If $M = M_1 \oplus \cdots \oplus M_s$ and $N = N_1 \oplus \cdots \oplus N_t$ are decompositions into indecomposable objects in $T\,\mathrm{grmod}$, then $s = t$, and there exist some permutation $\sigma \in \mathfrak{S}_s$ and integers $d_1, \ldots, d_s$ such that $N_i \cong M_{\sigma i}(d_i)$, where $-(d_i)$ denotes the shift of degree;*

(2) *$M \cong N$ as T-modules;*

(3) *$M_\mathfrak{m} \cong N_\mathfrak{m}$ as $T_\mathfrak{m}$-modules;*

(4) *$\hat{M} \cong \hat{N}$ as $\hat{T}$-modules.*

As an application, we compare the FFRT property of rings of characteristic p in the graded sense and in the local sense.

Corollary 3. *Let $R = \bigoplus_{i\geq 0} R_i$ be a Noetherian $\mathbb{Z}_{\geq 0}$-graded commutative ring such that $(R_0, \mathfrak{m}_0)$ is an F-finite Henselian local ring of prime characteristic p. Let $\mathfrak{m} = \mathfrak{m}_0 + R_+$, where $R_+ = \bigoplus_{i>0} R_i$. Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Let $M_1, \ldots, M_r$ be finitely generated $\mathbb{Q}$-graded R-modules. Then the following are equivalent.*

(1) *R has finite graded F-representation type (FFRT for short) by $M_1, \ldots, M_r$. That is,*

  (1-a) *For each i, M_i is indecomposable;*

  (1-b) *For each i, there exists some $e \geq 0$ and $c \in \mathbb{Q}$ such that $M_i(c)$ is a direct summand of ${}^{e}R$;*

(1-c) *For each $e \geq 0$, any indecomposable direct summand of ${}^eR$ is isomorphic to $M_i(c)$ for some $1 \leq i \leq r$ and $c \in \mathbb{Q}$.*

(2) *The local ring $\hat{R}$ has FFRT by $\hat{M}_1, \ldots, \hat{M}_r$. That is,*

(2-a) *For each i, $\hat{M}_i$ is indecomposable;*

(2-b) *For each i, there exists some $e \geq 0$ such that $\hat{M}_i$ is a direct summand of ${}^e\hat{R}$;*

(2-c) *For each $e \geq 0$, any indecomposable direct summand of ${}^e\hat{R}$ is isomorphic to $\hat{M}_i$ for some $1 \leq i \leq r$.*

In particular, R has FFRT in the graded sense if and only if $\hat{R}$ has FFRT.