SPECTRA OF TRIANGULATED CATEGORIES AND THEIR APPLICATIONS TO AFFINE AND PROJECTIVE VARIETIES

HIROKI MATSUI

This talk is based on joint work with Daigo Ito.

Let X be a quasi-projective variety. Here, a quasi-projective variety is an open subvariety of a projective variety. For example, affine and projective varieties are quasi-projective. In this talk, we consider $\mathrm{D}^{\mathrm{pf}}(X)$ the derived category of perfect complexes on X and consider the following question:

Question 0.1. Let X and Y be quasi-projective varieties. If $\mathrm{D}^{\mathrm{pf}}(X)$ and $\mathrm{D}^{\mathrm{pf}}(Y)$ are equivalent as triangulated categories, are X and Y isomorphic?

Balmer ([1]) showed that if the equivalence between $\mathrm{D}^{\mathrm{pf}}(X)$ and $\mathrm{D}^{\mathrm{pf}}(Y)$ preserves tensor products, then X and Y are isomorphic. Balmer's argument is as follows. For a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ (i.e., a triangulated category $\mathcal{T}$ equipped with a tensor product functor $\otimes$), he defined the ringed space

$$\mathrm{Spec}_{\otimes}(\mathcal{T}) := \{\mathcal{P} \mid \mathcal{P} \text{ is a prime thick ideal of } \mathcal{T}\},$$

where a thick subcategory $\mathcal{P} \subsetneq \mathcal{T}$ is a *prime thick ideal* if

- (ideal) for any $M \in \mathcal{T}$ and $N \in \mathcal{P}$, $M \otimes N \in \mathcal{P}$
- (prime) for any $M, N \in \mathcal{T}$, $M \otimes N \in \mathcal{P} \Rightarrow M \in \mathcal{P}$ or $N \in \mathcal{P}$.

For the tensor triangulated category $(\mathrm{D}^{\mathrm{pf}}(X), \otimes^{\mathbb{L}}_{\mathcal{O}_X}, \mathbf{1})$, it follows that $X \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X))$. Consequently, tensor preserving equivalence $\mathrm{D}^{\mathrm{pf}}(X) \simeq \mathrm{D}^{\mathrm{pf}}(Y)$ implies

$$X \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(Y)) \cong Y.$$

However, in general, triangulated equivalences between perfect derived categories do not preserve tensor structures. Therefore, extending Balmer's theory without using tensor structures is an important problem.

Recently, the author ([2, 3]) defined the ringed space

$$\mathrm{Spec}_{\triangle}(\mathcal{T}) := \{\mathcal{P} \mid \mathcal{P} \text{ is a prime thick subcategory of } \mathcal{T}\},$$

where a thick subcategory $\mathcal{P} \subsetneq \mathcal{T}$ is a *prime thick subcategory* if $\{\mathcal{X} \subseteq \mathcal{T} : \text{thick subcategory} \mid \mathcal{P} \subsetneq \mathcal{X}\}$ has a smallest element.

If $X = \mathrm{Spec}(R)$, Neeman ([4]) proved that there is an order-preserving bijection:

$$\left\{\begin{matrix}\text{specialization-closed subsets} \\ \text{of } \mathrm{Spec}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\text{thick ideals} \\ \text{of } \mathrm{D}^{\mathrm{pf}}(R)\end{matrix}\right\} = \left\{\begin{matrix}\text{thick subcategories} \\ \text{of } \mathrm{D}^{\mathrm{pf}}(R)\end{matrix}\right\}.$$

From this bijection, we obtain isomorphisms $\mathrm{Spec}(R) \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(R)) \cong \mathrm{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(R))$ of ringed spaces. On the other hand, if X is not affine, the inclusion

$$\left\{\begin{matrix}\text{thick ideals} \\ \text{of } \mathrm{D}^{\mathrm{pf}}(X)\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{thick subcategories} \\ \text{of } \mathrm{D}^{\mathrm{pf}}(X)\end{matrix}\right\}$$

is usually strict and hence the structure $\mathrm{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X))$ is more complicated.

The aim of this talk is to study the structure of $\mathrm{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X))$ by comparing it with $\mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \cong X$. The following result is the main theorem.

Theorem 0.2 ([5])**.** *Let X be a quasi-projective variety. Then there exists an open immersion*

$$X \cong \mathrm{Spec}_{\otimes}(\mathrm{D}^{\mathrm{pf}}(X)) \hookrightarrow \mathrm{Spec}_{\triangle}(\mathrm{D}^{\mathrm{pf}}(X))$$

of ringed spaces.

Using this theorem, we obtain several reconstruction results, which give answers to Question 0.1.

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THE FINITENESS OF DIMENSIONS AND RADII OF SUBCATEGORIES OF MODULES

YUKI MIFUNE

Let R be a commutative noetherian local ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules. The concepts of dimensions and radii of subcategories of modules have been introduced by Dao and Takahashi [4, 5]. These invariants measure the number of extensions required to generate a subcategory from a single object, up to direct summands, finite direct sums, and syzygies.

Dao and Takahashi [4] characterized a Cohen–Macaulay local ring with an isolated singularity by the finiteness of the dimension of the subcategory consisting of maximal Cohen–Macaulay modules that are locally free on the punctured spectrum. This result extends a classical result of Auslander, Huneke, Leuschke, and Wiegand [8], which establishes that finite Cohen–Macaulay representation type implies isolated singularity.

The following result suggests that the category of maximal Cohen–Macaulay modules generally has a finite dimension [6].

Theorem (Dey, Lank, and Takahashi)**.** *Let R be an excellent Cohen–Macaulay local ring with a canonical module. Then the dimension of the category of maximal Cohen–Macaulay modules is finite.*

On the other hand, there are few subcategories of finite dimension that are strictly contained in the category of maximal Cohen–Macaulay modules.

Theorem (Dao and Takahashi)**.** *Let R be a Cohen–Macaulay local ring.*

(1) *Let n be a nonnegative integer. Assume that the dimension of $\mathrm{CM}_n R$ is finite. Then the dimension of the singular locus of R is less than or equal to n. In particular, $\mathrm{CM}_n R$ coincides with $\mathrm{CM}\, R$.*
(2) *Assume that R is a local hypersurface. Then for any resolving subcategory $\mathcal{X}$ of $\operatorname{mod} R$ with $\operatorname{add} R \subsetneq \mathcal{X} \subsetneq \mathrm{CM}\, R$, the dimension of $\mathcal{X}$ is infinite.*

Here, the resolving subcategory is a class of subcategories that contains projective objects and is closed under direct summands, extensions, and kernels of epimorphisms. It was introduced by Auslander and Bridger [2] and has been thoroughly explored; see [1, 3, 7, 10, 11].

In this talk, we define the notion of the radius of two subcategories of modules, which is a common generalization of the dimension and radius of a subcategory introduced by Dao and Takahashi. The main results of this talk state the divergence of the radii of some specific subcategories, and in non-Cohen–Macaulay case, the category $\mathrm{C}(R)$, considered as a counterpart to the category of maximal Cohen–Macaulay modules, is more likely to be infinite-dimensional.

This talk is based on a preprint [9].

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ON LOCAL RINGS OF FINITE SYZYGY REPRESENTATION TYPE

YUYA OTAKE AND KAITO KIMURA

Auslander [1] proved that a Cohen–Macaulay complete local ring R has an isolated singularity if R has finite Cohen–Macaulay representation type. This result is a fundamental theorem in Cohen–Macaulay representation theory [5, 6] and has been expanded to excellent Cohen–Macaulay rings by Huneke and Leuschke [3]. Recently, Dao and Takahashi [2] showed that for a Cohen–Macaulay local ring R, if the category of finitely generated maximal Cohen–Macaulay local rings which are locally free on the punctured spectrum of R has finite radius, then R has an isolated singularity, and the converse is also true if R is complete, equicharacteristic and with perfect residue field. In this talk, we consider the finiteness of dimension of the category of higher syzygy modules over an arbitrary noetherian local ring.

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Prismatic Kunz's theorem

Ryo Ishizuka (Institute of Science Tokyo)*

This talk is based on joint work with Kei Nakazato [IN24]. Fix a prime number p. In positive characteristic commutative algebra, Kunz's theorem characterizes regular rings by the faithfulness of the Frobenius map:

Theorem (Kunz's theorem [Kun69]). *Let R be a Noetherian ring of positive characteristic p. The following conditions are equivalent:*

1. *R is a regular ring.*
2. *The Frobenius map $F : R \to R$ is faithfully flat.*
3. *The canonical map $R \to R_{\mathrm{perf}} := \operatorname{colim}_F R$ is faithfully flat.*

This is a starting point for the study of singularities in positive characteristic by using the Frobenius map. Bhatt–Iyengar–Ma generalized Kunz's theorem by focusing on the perfectness of the perfect closure R_{perf} instead of the Frobenius map. They obtained the following p-adic version of Kunz's theorem:

Theorem (p-adic Kunz's theorem [BIM19]). *Let R be a Noetherian ring whose Jacobson radical contains p. The following conditions are equivalent:*

1. *R is a regular ring.*
2. *There exists a perfectoid ring A and a faithfully flat ring map $R \to A$.*

Very recently, some researchers have been trying to generalize positive characteristic singularities (e.g., F-pure singularity) to mixed characteristic using perfectoid rings [BMP+24] as in the p-adic Kunz's theorem. On the other hand, it was not clear how to generalize Kunz's theorem using the "Frobenius map". Recently, Bhatt pointed out that the "Frobenius map" can be generalized to mixed characteristic using the prismatic theory introduced by Bhatt–Scholze [BS22]:

Slogan (cf. [Bha22, Remark 5.6]). *Let R be a "good" Noetherian local ring of residue characteristic p. The "mixed characteristic Frobenius map" on R is the Frobenius lift φ on $\mathbb{\Delta}_{R/A}$, the* prismatic cohomology *of R as an A/I-algebra with a prism (A, I).*

In this context, we proved a mixed characteristic version of Kunz's theorem using the "mixed characteristic Frobenius map":

Main Theorem ([IN24]). *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring of residue characteristic p. Then there exists a prism (A, I) and a surjective ring map $A \twoheadrightarrow R$ such that A is an unramified complete regular local ring and* $\dim(A) = \operatorname{emdim}(R) + 1$. *The following conditions are equivalent:*

1. *R is a regular local ring.*
2. *The Frobenius lift $\varphi\colon \mathbb{\Delta}_{R/A} \to \varphi_{A,*}\mathbb{\Delta}_{R/A}$ of the δ-A-algebra $\mathbb{\Delta}_{R/A}$ is faithfully flat.*

In this talk, we introduce p-adic Kunz's theorem by Bhatt–Iyengar–Ma and explain our main theorem.

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EXAMINING KEMPE EQUIVALENCE VIA COMMUTATIVE ALGEBRA

AKIYOSHI TSUCHIYA

This talk is based on joint work with Hidefumi Ohsugi [3].

A *k-coloring* f of a graph G on the vertex set $[d] := \{1,2,\dots,d\}$ is a map from $[d]$ to $[k]$ such that $f(i) \neq f(j)$ for all $\{i,j\} \in E(G)$. The smallest integer $\chi(G)$ such that G has a $\chi(G)$-coloring is called the *chromatic number* of G. Given a k-coloring f of G, and integers $1 \le i < j \le k$, let H be a connected component of the induced subgraph of G on the vertex set $f^{-1}(i) \cup f^{-1}(j)$. Then we can obtain a new k-coloring g of G by setting

$$g(x) = \begin{cases} f(x) & x \notin H, \\ i & x \in H \text{ and } f(x) = j, \\ j & x \in H \text{ and } f(x) = i. \end{cases}$$

We say that g is obtained from f by a *Kempe switching*. Two k-colorings f and g of G are called *Kempe equivalent*, denoted by $f \sim_k g$, if there exists a sequence $f_0, f_1, \dots, f_s$ of k-colorings of G such that $f_0 = f$, $f_s = g$, and f_i is obtained from f_{i-1} by a Kempe switching. Let denote $\mathscr{C}_k(G)$ the set of all k-colorings of G. Then $\sim_k$ is an equivalence relation on $\mathscr{C}_k(G)$. The equivalence classes of $\mathscr{C}_k(G)$ by $\sim_k$ are called the k-Kempe classes. We denote $\mathrm{kc}(G,k)$ the quotient set $\mathscr{C}_k(G)/\sim_k$ and denote $\mathrm{Kc}(G,k)$ the number of k-Kempe classes of G, namely $\mathrm{Kc}(G,k) = |\mathrm{kc}(G,k)|$. Kempe switchings have been introduced by Kempe in the false proof of the 4-Color Theorem. However, his idea is powerful in graph coloring theory. Recently, many researchers have studied Kempe switchings and Kempe equivalence. See [1] for an overview of Kempe equivalence.

Let G be a graph on the vertex set $[n]$ with the edge set $E(G)$. Given a subset $S \subset [n]$, let $G[S]$ denote the induced subgraph of G on the vertex set S. A subset $S \subset [n]$ is called a *stable set* (or an *independent set*) of G if $\{i,j\} \notin E(G)$ for all $i,j \in S$ with $i \neq j$. Namely, a subset $S \subset [n]$ is stable if and only if $G[S]$ is an empty graph (i.e., a graph with no edges). In particular, the empty set $\emptyset$ and any singleton $\{i\}$ with $i \in [n]$ are stable. Denote $S(G) = \{S_1, \dots, S_m\}$ the set of all stable sets of G. Given a subset $S \subset [n]$, we associate the $(0,1)$-vector $\rho(S) = \sum_{j \in S} \mathbf{e}_j$. Here $\mathbf{e}_j$ is the jth unit coordinate vector in $\mathbb{R}^n$. For example, $\rho(\emptyset) = (0,\dots,0) \in \mathbb{R}^n$. Let $K[\mathbf{t},s] := K[t_1,\dots,t_n,s]$ be the polynomial ring in $n+1$ variables over a field K. Given a nonnegative integer vector $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{Z}^n_{\ge 0}$, we write $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n} \in K[\mathbf{t},s]$. The *stable set ring* of G is

$$K[G] := K[\mathbf{t}^{\rho(S_1)}s, \dots, \mathbf{t}^{\rho(S_m)}s] \subset K[\mathbf{t},s].$$

We regard $K[G]$ as a homogeneous algebra by setting each $\deg(\mathbf{t}^{\rho(S_i)}s) = 1$. Note that $K[G]$ is a toric ring. Let $K[\mathbf{x}] := K[x_1,\dots,x_m]$ denote the polynomial ring in m variables over K with each $\deg(x_i) = 1$. The *stable set ideal* I_G of G is the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by $\pi(x_i) = \mathbf{t}^{\rho(S_i)}s$ for $1 \le i \le m$. Note that I_G is a toric ideal, and generated by homogeneous binomials. The toric ring $K[G]$ is called *quadratic* if I_G is generated by quadratic binomials. We say that "I_G is generated by quadratic binomials" even if $I_G = \{0\}$ (or equivalently, G is complete). It is easy to see that a homogeneous binomial $x_{i_1} \cdots x_{i_r} - x_{j_1} \cdots x_{j_r} \in K[\mathbf{x}]$ belongs to I_G if and only if $\bigcup_{\ell=1}^{r} S_{i_\ell} = \bigcup_{\ell=1}^{r} S_{j_\ell}$ as multisets. In [2], the authors showed that I_G is generated by binomials $\mathbf{x}_f - \mathbf{x}_g$ associated with k-colorings f and g of a replication graph of an induced subgraph of G, and found a relationship between Kempe equivalence on G and an algebraic property of I_G. In particular, by using the proof of [2, Theorem 1.3], we can examine if two k-colorings of G are Kempe equivalent by using I_G. However, I_G has too much information for this purpose. In this talk, we introduce a simpler ideal J_G, which is generated by binomials $\mathbf{x}_f - \mathbf{x}_g$ associated with 2-colorings f and g of an induced subgraph of G, to determine Kempe equivalence on G. We call J_G the 2-*coloring ideal* of G. Then our first main result is the following:

Theorem 0.1. *Let G be a graph on $[d]$ and let f, g be k-colorings of G. Then $f \sim_k g$ if and only if $\mathbf{x}_f - \mathbf{x}_g \in J_G$.*

Next, we compute all k-colorings of a graph G up to Kempe equivalence by virtue of the algebraic technique on Gröbner bases. Namely, a complete representative system for $\mathrm{kc}(G,k)$ is given. For this, we introduce another ideal K_G defined by

$$K_G := J_G + M_G,$$

where

$$M_G := \langle x_S x_T \mid S, T \in S(G), S \cap T \neq \emptyset \rangle.$$

The ideal K_G is called the *Kempe ideal* of G. Then our second main result is the following:

Theorem 0.2. *Let G be a graph on $[d]$ and $<$ a monomial order on $R[G]$, and let $\{\mathbf{x}_{f_1}, \dots, \mathbf{x}_{f_s}\}$ be the set of all standard monomials of degree k with respect to the initial ideal $\mathrm{in}_<(K_G)$. Then*

$$\{f_1, \dots, f_s\} \cap \mathscr{C}_k(G)$$

is a complete representative system for $\mathrm{kc}(G,k)$.

As a consequence, the number of k-Kempe classes $\mathrm{Kc}(G,k)$ can be computed by Hilbert functions.

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A LINEAR VARIANT OF THE NEARLY GORENSTEIN PROPERTY

SORA MIYASHITA

1. Nearly Gorenstein rings and its mild generalization on certain rings

We refer to R as a graded ring with the assumption that $R = \bigoplus_{n\geq 0} R_n$ and $R_0 = \Bbbk$, where $\Bbbk$ is a field. Additionally, we assume R is Cohen-Macaulay and admits a canonical module ω_R. Define the **trace** of ω_R as $\mathrm{tr}_R(\omega_R) := \sum_{\phi\in \mathrm{Hom}_R(\omega_R,R)} \phi(\omega_R)$. Let $\mathfrak{m}_R$ denote the unique graded maximal ideal of R.

Definition 1 ([1])**.** If $\mathrm{tr}_R(\omega_R) \supset \mathfrak{m}_R$, we say that R is **nearly Gorenstein**.

A graded ring R is **standard graded** if $R = \Bbbk[R_1]$, and **semi-standard graded** if R is finitely generated $\Bbbk[R_1]$-module. The author introduced the following condition.

Definition 2 ([2])**.** Let R be a Cohen–Macaulay semi-standard graded ring. We say that R **satisfies** ($\natural$) if $\sqrt{(\mathrm{tr}_R(\omega_R))_1 R} \supset \mathfrak{m}_R$. Note that the following hold:

R is nearly Gorenstein $\Rightarrow$ R satisfies ($\natural$) $\Rightarrow$ R is Gorenstein on the punctured spectrum.

2. Main results

Given positive integer $k > 0$, the subring $R^{(k)} := \bigoplus_{n\geq 0} R_{kn}$ is called the k**-th Veronese subring** of R. Let $a(R)$ denote the a**-invariant** of R. The purpose of this talk is to explain the following two statements via some concrete examples.

Theorem 3. *Let R be a Cohen–Macaulay semi-standard graded ring satisfying ($\natural$). Then the following statements hold:*

1. *$R^{(k)}$ satisfies ($\natural$) for any $k > 0$.*
2. *Assume R is standard graded with $\dim R > 0$. Then the following is true:*
   - (a) *If R satisfies ($\natural$), then $R^{(k)}$ is nearly Gorenstein for any $k > a(R/\mathrm{tr}_R(\omega_R)_1 R)$;*
   - (b) *If R is nearly Gorenstein, then $R^{(k)}$ is nearly Gorenstein for any $k > 0$.*

Theorem 4. *Let R be a Cohen–Macaulay semi-standard graded* **domain** *(or* **level ring***). If R satisfies ($\natural$), there exists $k_R > 0$ such that $R^{(k)}$ is nearly Gorenstein for all $k \geq k_R$.*

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