Asymptotics for the focusing integrable discrete nonlinear Schrödinger equation

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1. Nonlinear Schrödinger equation and a soliton

focusing NLS

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

soliton

$$u(x,t) = 2\eta e^{2i\xi x - 4i(\xi^2 - \eta^2)t + i(\psi_0 + \pi/2)} \times \operatorname{sech}(2\eta x - 8\xi\eta t - 2\delta_0)$$

carrier wave ($\exp$, oscillatory) × traveling solitary wave (sech)

2. Long-time asymptotics of soliton equations

As $t \to \infty$, the solution is asymptotically
a sum of solitons plus a small perturbation.

NLS: Fokas-Its '96 (IBVP), Kamvissis '95 (IVP)
Toda lattice: Krüger-Teschl '09 (IVP)
KdV: Tanaka '75 (IVP), Grunert-Teschl '09 (IVP)

SOLITON RESOLUTION in recent terminology
(e.g. Terence Tao's "Why are solitons stable?", 2009)
Valid for non-integrable equations as well,
but INTEGRABLE ones are particularly important because

- they are model cases
- phase shift can be written down in the inverse scattering parlance.

3. Integrable Discrete NLS (IDNLS) 1

NLS (**focusing**)

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

Ablowitz-Ladik ('75)
integrable discrete nonlinear Schrödinger equation (**focusing**)

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) + |R_n|^2(R_{n+1} + R_{n-1}) = 0$$

Both have solitons:
carrier wave ($\exp$, oscillatory) $\times$ traveling solitary wave (sech)

4. Integrable Discrete NLS (IDNLS) 2

$R_n(t) = \mathrm{BS}\,(n,t;z_1,C_1(0))$, soliton

$z_1 = \exp(\alpha_1 + i\beta_1)$, $\alpha_1 > 0$: eigenvalue

$C_1(0)$:norming constant (at $t = 0$)

$$\begin{aligned}\mathrm{BS}(n,t;z_1,C_1(0)) &= \text{carrier wave} \times \text{traveling wave} \\ &= \exp\big(-i[2\beta_1(n+1) - 2w_1 t - \arg C_1(0)]\big) \\ &\quad \times \sinh(2\alpha_1)\mathrm{sech}[2\alpha_1(n+1) - 2v_1 t - \boldsymbol{\theta_1}].\end{aligned}$$

v_1, w_1: written in terms of α_1, β_1.

$\boldsymbol{\theta_1}$: written in terms of $|C_1(0)|, \alpha_1$.

If we multiply $C_1(0)$ by another number

$\Rightarrow$ PHASE SHIFT in $\exp$ and sech.

It happens when solitons with different velocities collide.

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5. IDNLS and its Lax pair

$$i\frac{d}{dt}R_n+(R_{n+1}-2R_n+R_{n-1})+|R_n|^2(R_{n+1}+R_{n-1})=0 \quad \text{(IDNLS)}$$

. . . .

n-part : $X_{n+1}=\begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n$

t-part : $\dfrac{d}{dt}X_n=\Bigg[\text{a complicated matrix}\Bigg]X_n$

(IDNLS) is the compatibility condition.

6. Eigenfunctions of the n-part

If $R_n \to 0$ (rapidly) as $n \to \pm\infty$, then approximately

$$X_{n+1} \approx \begin{bmatrix} z & \boxed{0} \\ \boxed{0} & z^{-1} \end{bmatrix} X_n.$$

'solutions' $\begin{bmatrix} z^n \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ z^{-n} \end{bmatrix}$

There exist eigenfunctions
$\phi_n(z), \psi_n(z)$ in $|z| \geq 1$ and $\psi_n^*(z)$ in $|z| \leq 1$
which behave like $z^{\pm n}$ as $n \to \pm n$.

3 solutions in the 2-dimensional solution space.
(There's another, but we omit it.)

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7. Eigenvalues and the reflection coefficient

On $|z| = 1$, $\exists a(z)$, $b(z) = b(z,t)$ such that

$$\boxed{\phi_n(z) = b(z)\psi_n(z) + a(z)\psi_n^*(z),}$$

If $\boxed{a(z_j) = 0}$, then $a(-z_j) = 0$.

$\{\pm z_j, \pm \bar{z}_j^{-1}\}$ is called a $\boxed{\text{quartet of eigenvalues}}$.

It corresponds to a $\boxed{\text{soliton}}$.

On $|z| = 1$, $\boxed{\text{the reflection coefficient } r(z)}$ is

$$r(z) := \frac{b(z)}{a(z)}$$

8. Scattering data

Assume $a(z_j) = 0$ (order 1). $\pm z_j$ is an eigenvalue.
$\phi_n(z_j) = \exists b_j \psi_n(z_j)$.

The norming constant is defined by $C_j := \dfrac{b_j}{\frac{d}{dz} a(z_j)}$

Scattering Data

$$\{(\pm z_j, \pm \bar{z_j}^{-1}, C_j)\}_{j=1}^{J}, \qquad r(z)$$

The potential R_n is said to be reflectionless if $r(z) = 0$.

Inverse Scattering Transform

The potential R_n is reconstructed from the scattering data.
Done by using a Riemann-Hilbert problem with poles.

9. Initial Value Problem

(Simple) Time Evolution of the Scattering Data

Eigenvalues are independent of time.

$C_j(t) = C_j(0) \exp\left(it(z_j - z_j^{-1})^2\right),$

$r(z,t) = r(z) \exp\left(it(z - z^{-1})^2\right)$ on $|z| = 1,$

where $r(z) := r(z,0)$

Initial Value Problem

Initial value $R_n(0)$ determines the scattering data at $t = 0$.

The scattering data at $t > 0$ can be calculated.

Potential Reconstruction (RHP). $R_n(t)\,(t > 0)$ obtained.

10. Reflectionless Case

If $\boxed{\boxed{r(z) = r(z,0) = 0}}$,

$R_n(t) =$ multi-soliton.

It approaches a sum of 1-solitons as $t \to \infty$.

$\boxed{\boxed{\text{phase shift}}}$ (formal proof in Ablowitz-Prinari-Trubatch '04)

Each term is of the form $\mathrm{BS}(n, t, z_j, \underbrace{p_j T(z_j)^{-2}}_{\text{phase shift}} C_j(0))$

Phase shift is determined by the eigenvalues:

$$p_j := \prod_{k>j} z_k^2 \bar{z}_k^{-2}, \qquad T(z_j) := \prod_{k>j} \frac{z_k^2(z_j^2 - \bar{z}_k^{-2})}{z_j^2 - z_k^{-2}}$$

The j-th soliton is faster than the $(j-1)$-th.

11. Main results (sketched)

What happens as $t \to \infty$ if there *is* reflection $(r(z) \not\equiv 0)$?

SOLITON RESOLUTION

A sum of 1-solitons plus a small perturbation

A new PHASE SHIFT formula involving the REFLECTION COEFFICIENT $r(z)$.

$\boxed{|n|/t < 2 \text{ (the 'timelike' region)}}$

There is a new factor written in terms of $r(z)$:

$\mathrm{BS}(n, t, z_j,\ \text{New} \cdot p_j T(z_j)^{-2} C_j(0))$

$\boxed{|n|/t \geq 2}$

Leading term is the same as in the reflectionless case.

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12. Asymptotic Behavior: $r(z) \neq 0$

♣ **tw** is the velocity of the soliton (traveling wave).

$\boxed{|\mathrm{tw}(z_j)| < 2}$ *Timelike Region: New Phase Shift Formula*

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, \boldsymbol{\delta(0)\delta(z_j)^{-2}} p_j T(z_j)^{-2} C_j(0)\right) + O(t^{-1/2}).$$

$\boldsymbol{\delta(z)}$ determined by $r(z)$.
p_j, $T(z_j)$ determined by z_k's $(k \geq j)$.
z_k's correspond to the j-th and faster solitons.

$\boxed{|\mathrm{tw}(z_j)| = 2}$ Leading term remains the same

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, p_j T(z_j)^{-2} C_j(0)\right) + O(t^{-1/3}).$$

$\boxed{|\mathrm{tw}(z_j)| > 2}$ Leading term remains the same

As $|n| \to \infty$,

$$R_n(t) = \mathrm{BS}\left(n, t; z_j, p_j T(z_j)^{-2} C_j(0)\right) + O(n^{-k}), \quad \forall k.$$

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13. Riemann-Hilbert Problems (RHPs)

Γ: oriented contour (the left-hand is the $+$ side).
$m(z)$: unknown matrix, holomorphic in $\mathbb{C} \setminus \Gamma$
$m_\pm$: boundary values on Γ from the $\pm$ sides

$\boxed{\boxed{\textit{RHP: } m_+ = m_- v \textit{ on } \Gamma}}$ (v : **the jump matrix**)

Γ can be DEFORMED.

RHP with poles

If $m(z)$ admits poles, we impose conditions on its residues.

Inverse scattering

1. Jump matrix written in terms of the reflection coefficient.
2. Poles are eigenvalues.
3. Residue conditions written in terms of norming constants.

14. Inv. Scatt./RHP for the focusing IDNLS

$m_+(z) = m_-(z)v(z)$ on $|z| = 1$(clockwise),

$$v(z) = \begin{bmatrix} 1 + |r(z)|^2 & e^{-2\varphi}\bar{r}(z) \\ e^{2\varphi} r(z) & 1 \end{bmatrix}$$ jump matrix

$$\varphi = \frac{1}{2} it(z - z^{-1})^2 - n \log z$$ *phase function!*

$$\mathrm{Res}(m(z); \pm z_j) = \lim_{z \to \pm z_j} m(z) \begin{bmatrix} 0 & 0 \\ z_j^{-2n} C_j(t) & 0 \end{bmatrix}, \ \forall j,$$

$$\mathrm{Res}(m(z); \pm \bar{z}_j^{-1}) = \lim_{z \to \pm \bar{z}_j^{-1}} m(z) \begin{bmatrix} 0 & \bar{z}_j^{-2n-2} \bar{C}_j(t) \\ 0 & 0 \end{bmatrix}, \ \forall j.$$

$m(z) \to I$ as $z \to \infty$,

Potential Reconstruction $$R_n(t) = -\frac{d}{dz} m(z)_{21} \Big|_{z=0}$$

IVP solved!

15. Different behaviours in different regions, why?

RHP admits contour deformation.

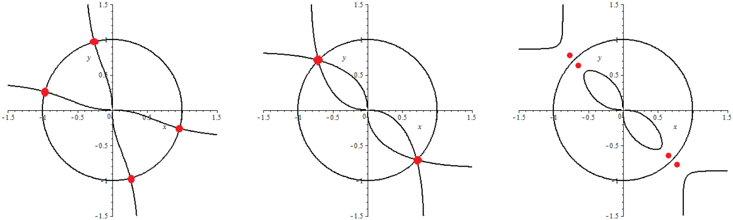
NONLINEAR STEEPEST DESCENT (Deift-Zhou).

$$\varphi = \frac{1}{2} it(z - z^{-1})^2 - n \log z \quad \text{phase function}$$

Geometry of saddle (stationary) points plays an important role.

The curve $\boxed{\operatorname{Im} \varphi(z) = 0}$ and the saddle points or the stationary points of higher order.

$|n|/t < 2$(saddle) $\qquad$ $|n|/t = 2$ $\qquad$ $|n|/t > 2$(saddle)



Thank you very much.

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