Games & Algorithms with Hook Structures

Shoji Conference, March 2012

Noriaki KAWANAKA

Most part of this talk

to appear in :

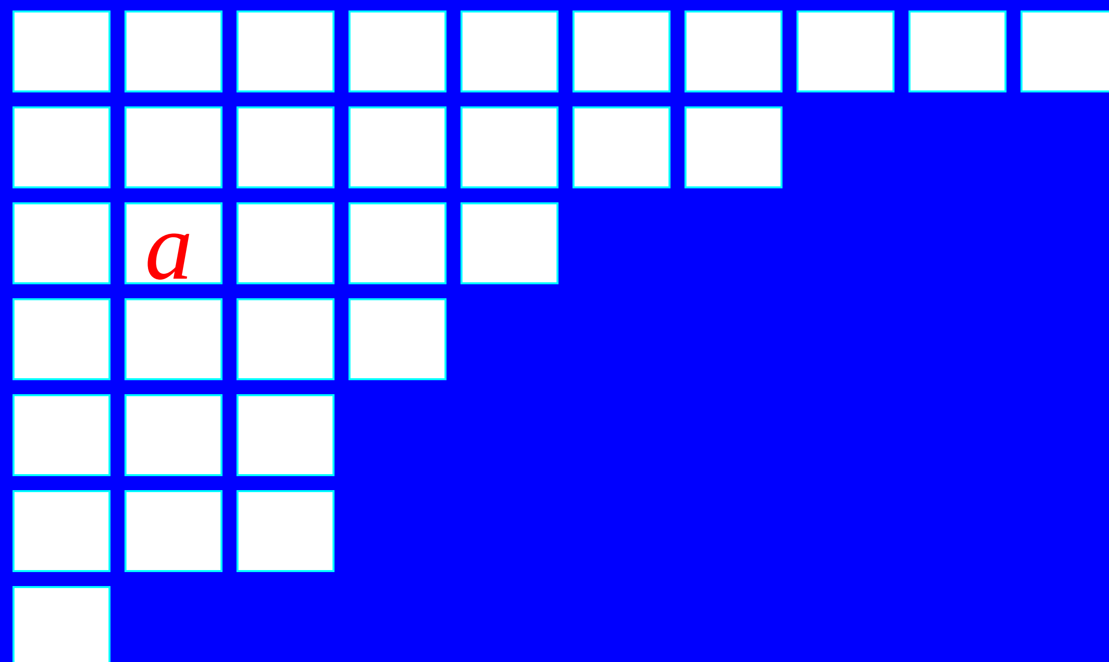
Sugaku Expositions (AMS)

(appeared in :

Sugaku (Iwanami))

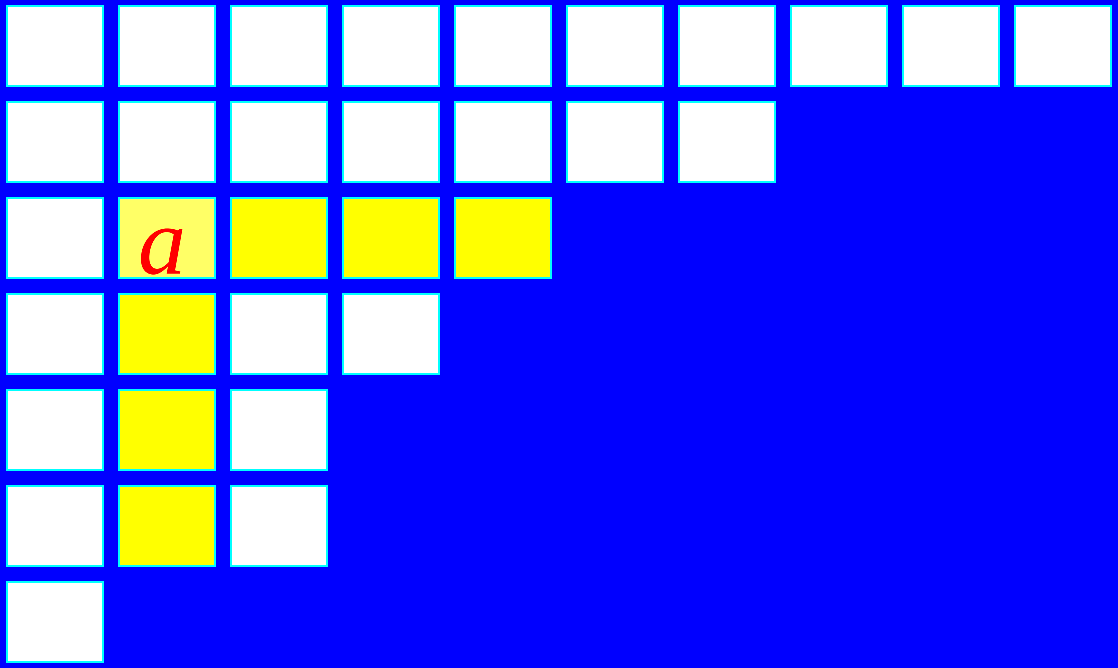
Young diagram   Y

Y =



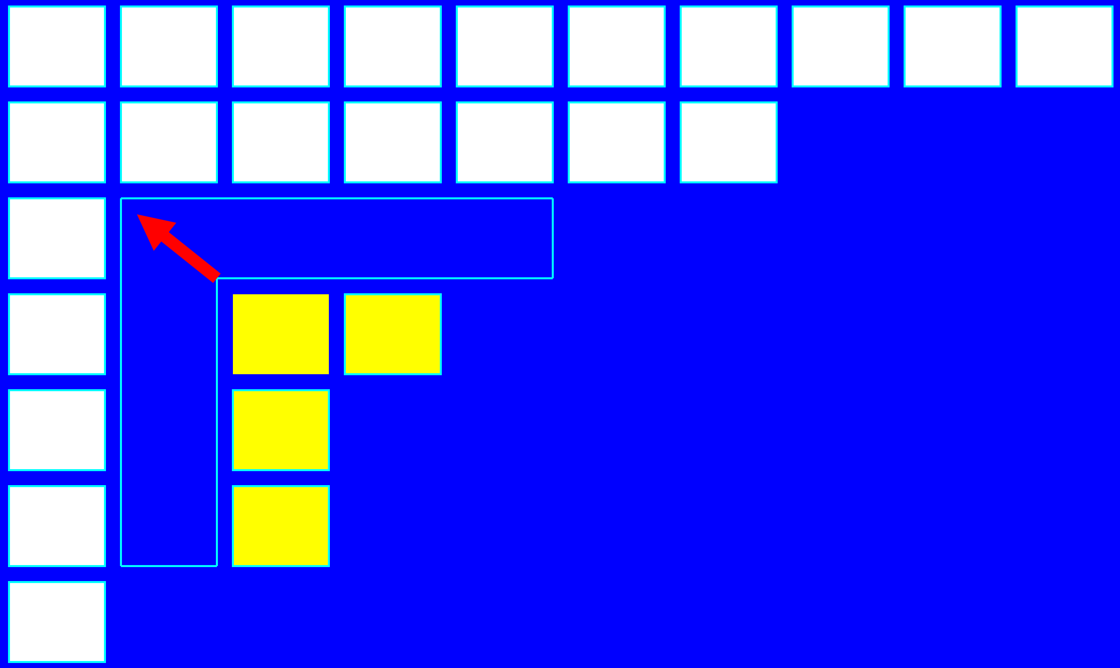
the hook H(a) of $a \in$ Y

Y =



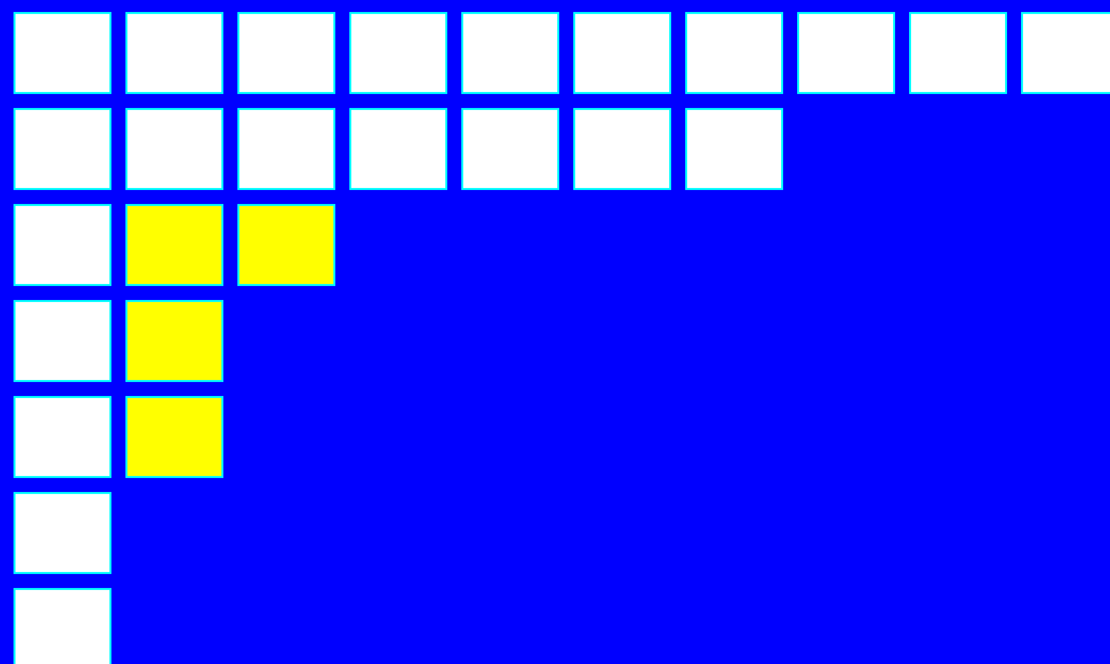
Subtraction Y−H(a) (1)

Y−H(a) =



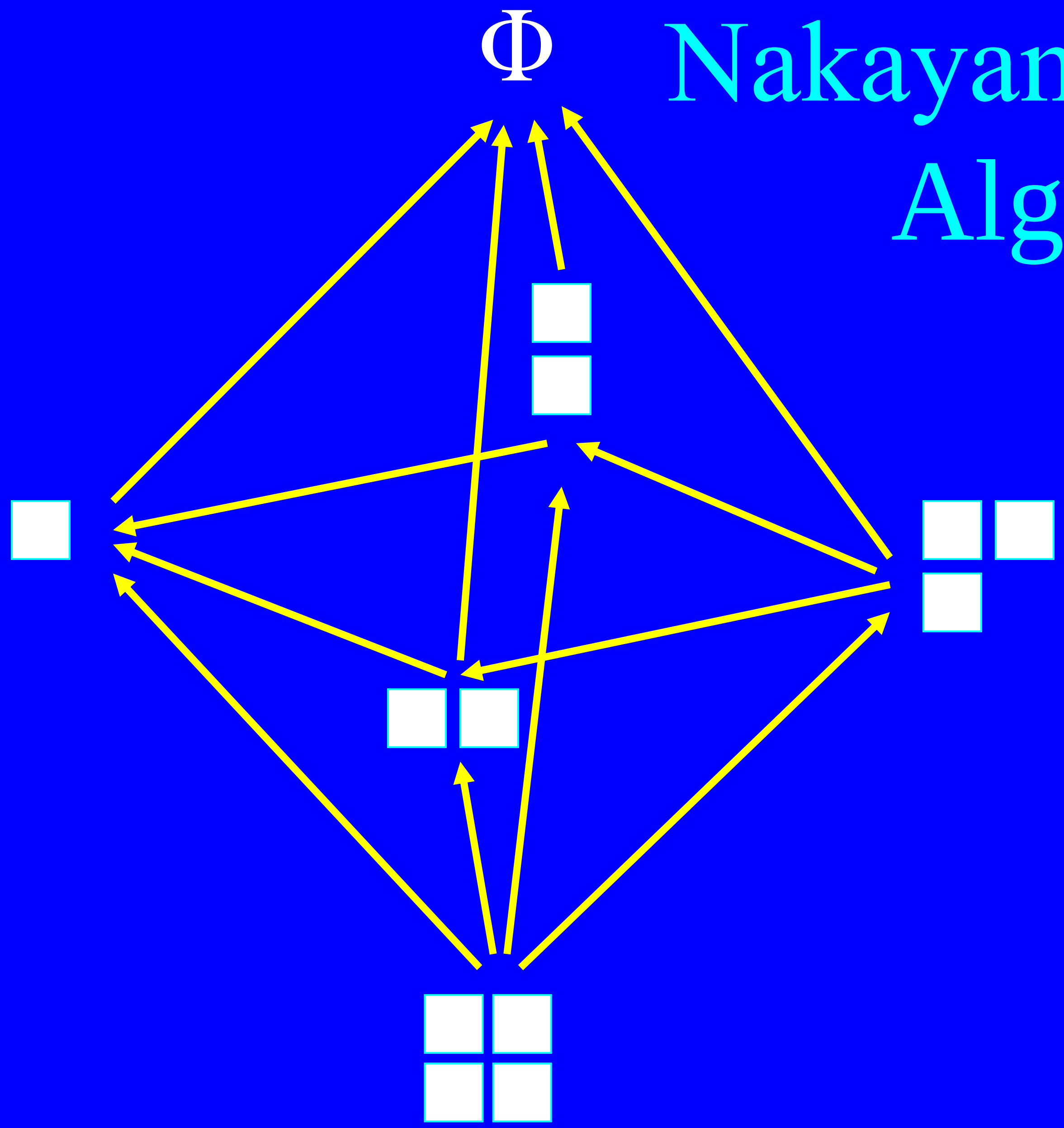
Subtraction Y − H(a) (2)

Y − H(a) =



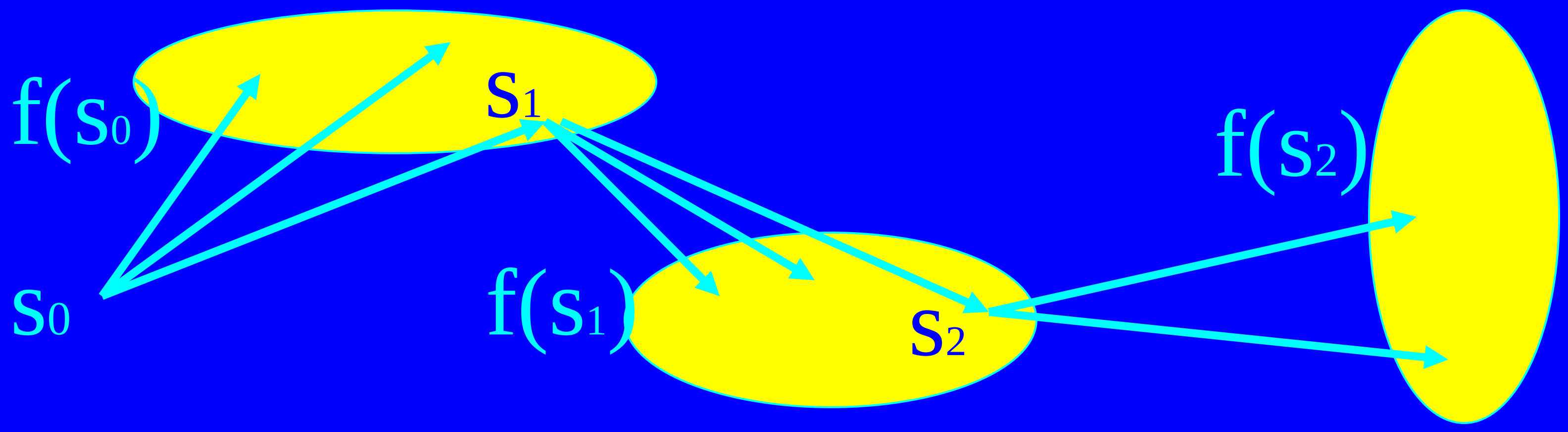
Introduced by Nakayama 1940

Nakayama's Algorithm



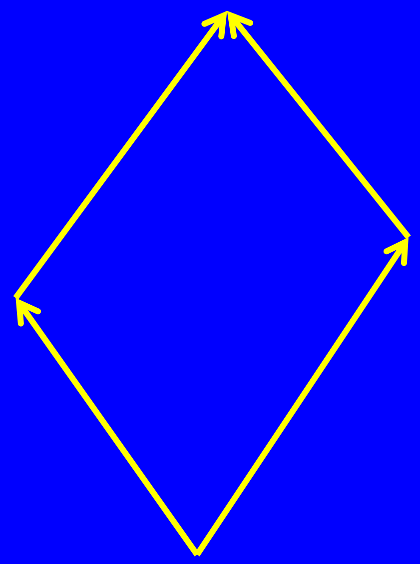
(S, f) = (abstract) algorithm

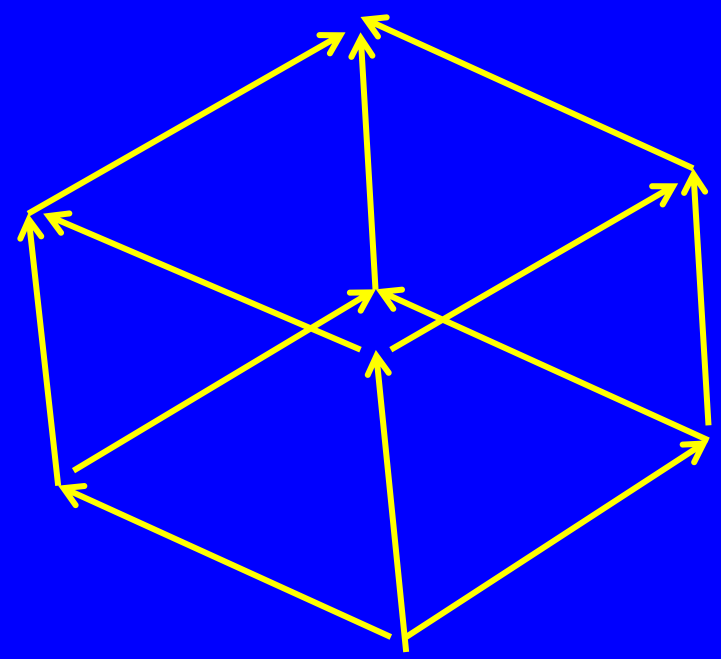
S : set , $f : S \to 2^S$



n-cube

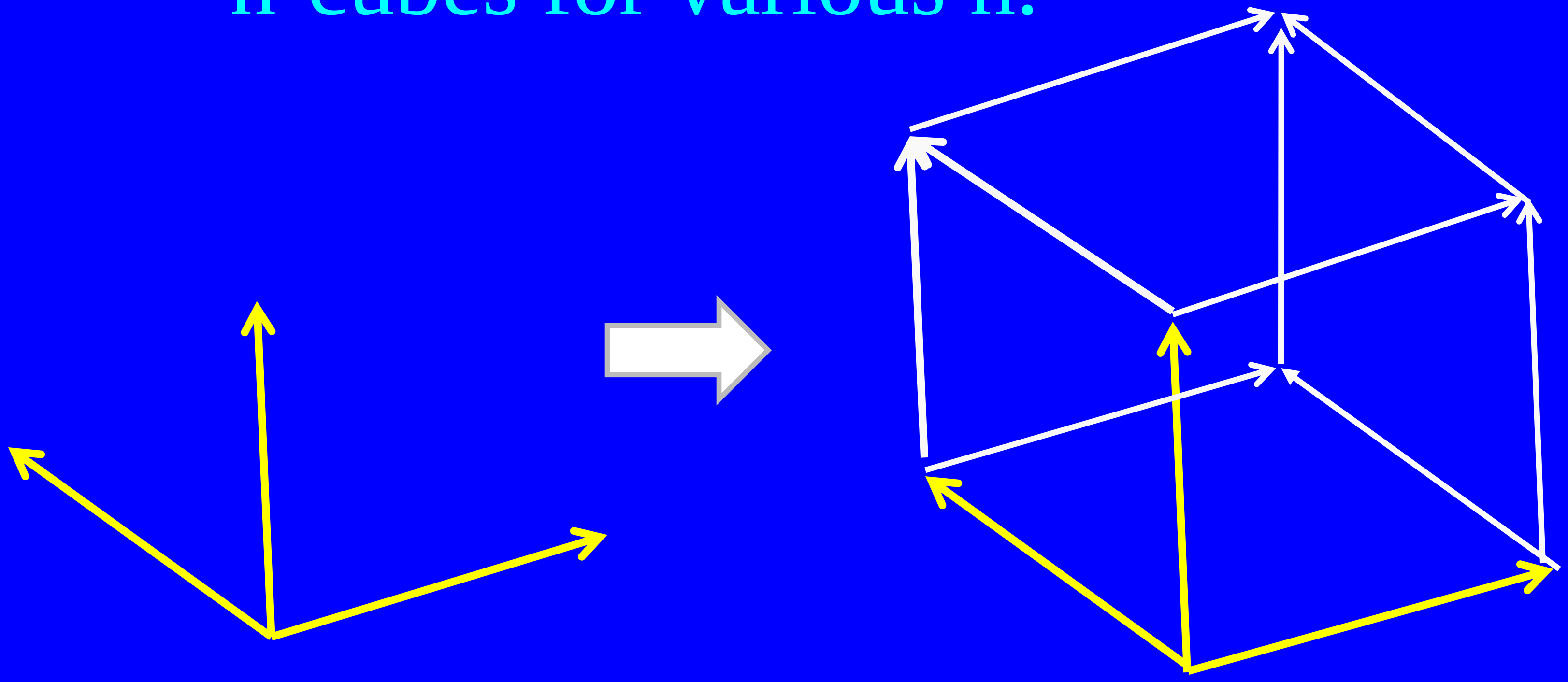
square = 2-cube   3-cube · · ·





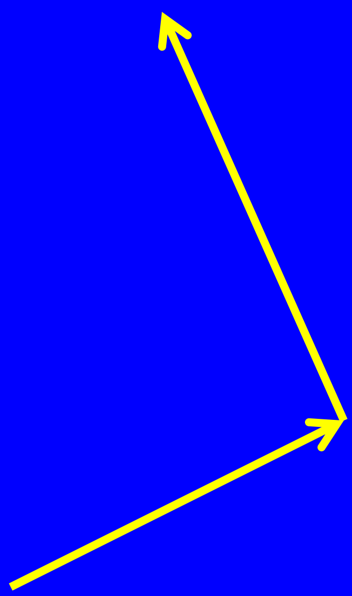
Plain algorithm (1)

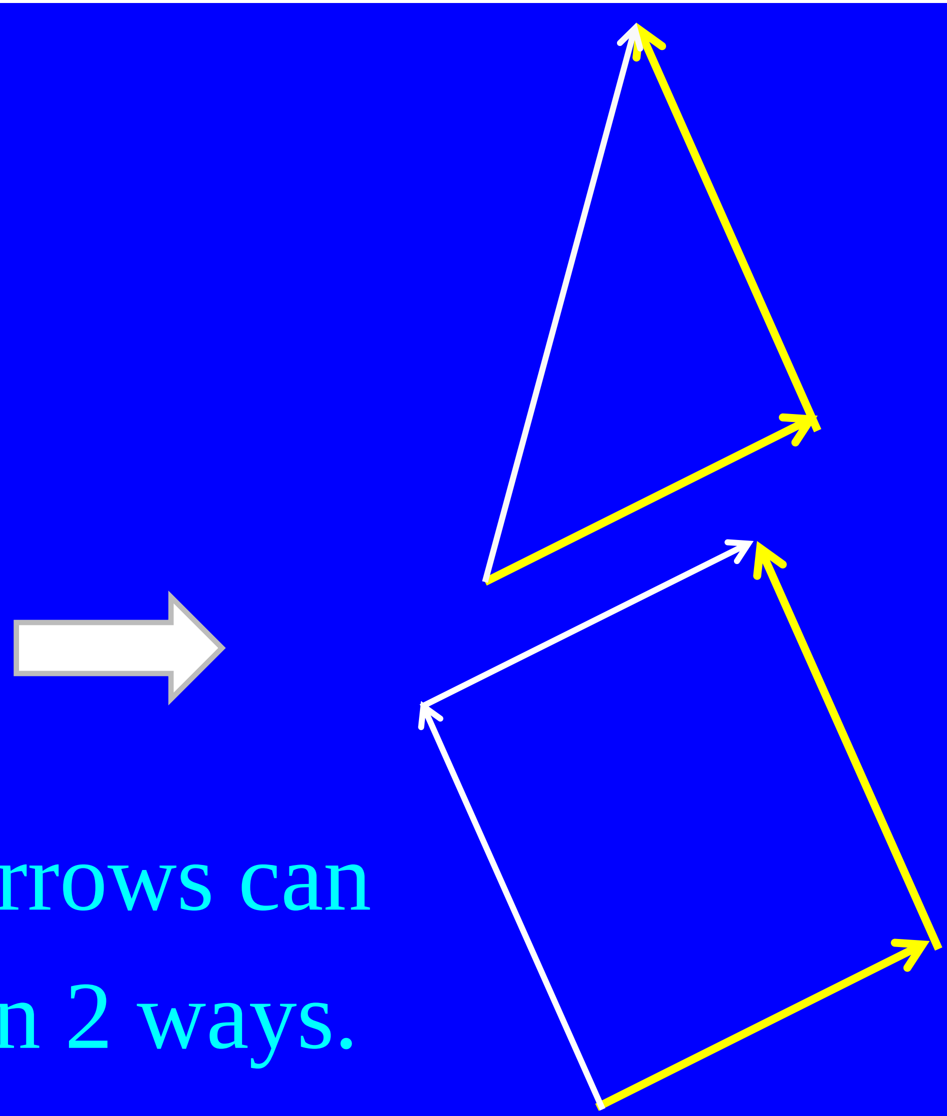
(P1) A plain algorithm contains a lot of n-cubes for various n.



Plain algorithm (2)

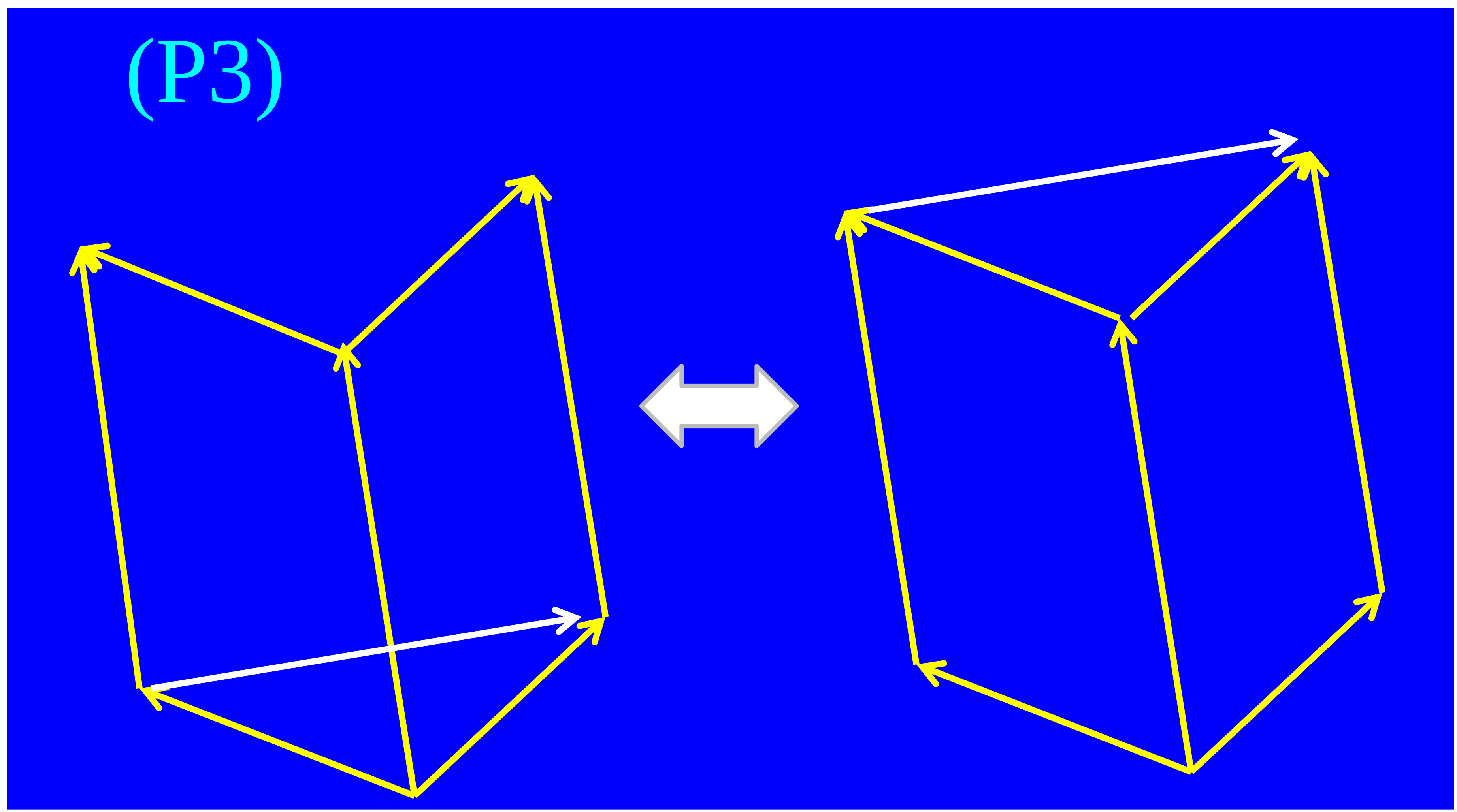
(P2)



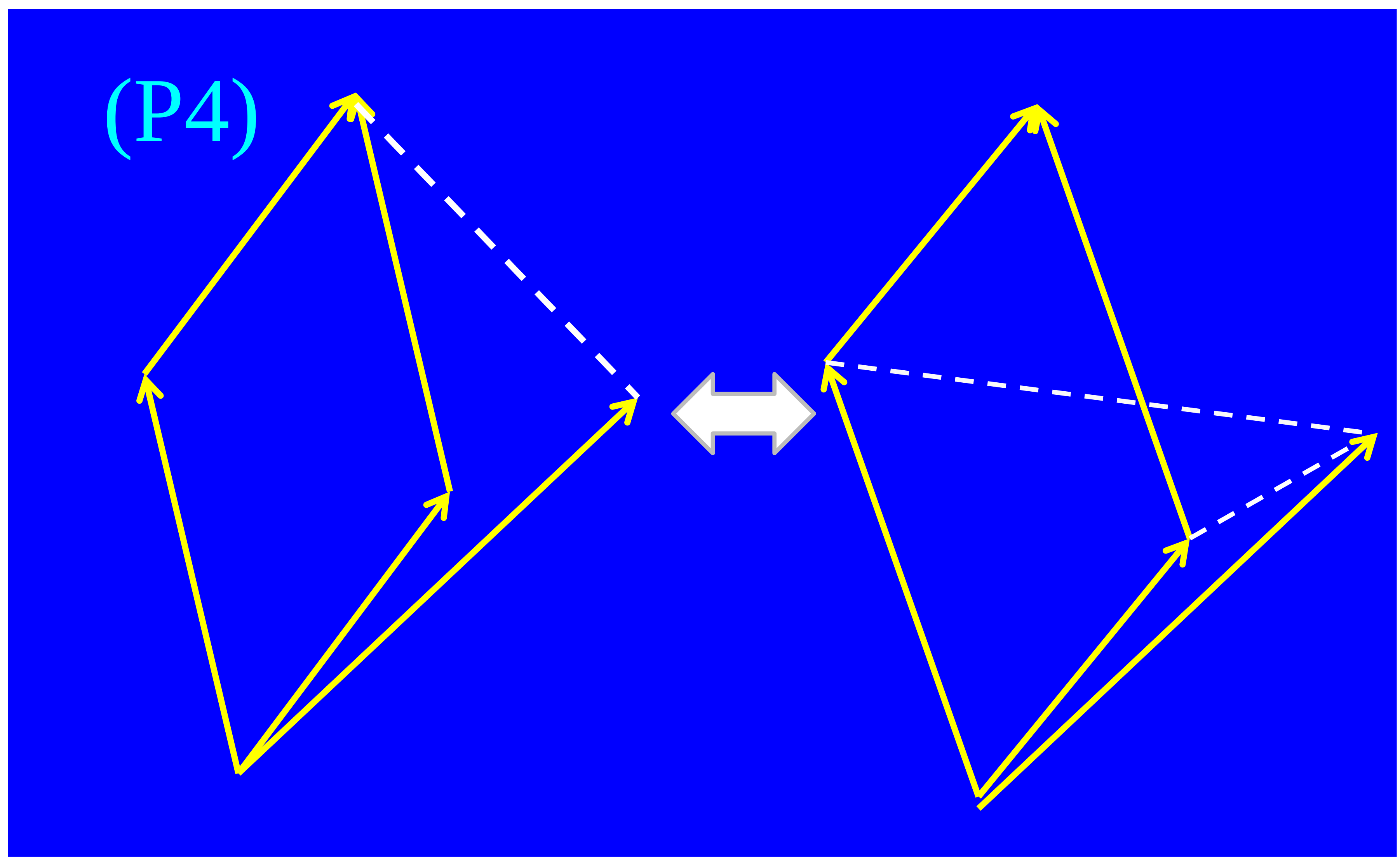


2 consecutive arrows can be completed in 2 ways.

Plain algorithm (3)

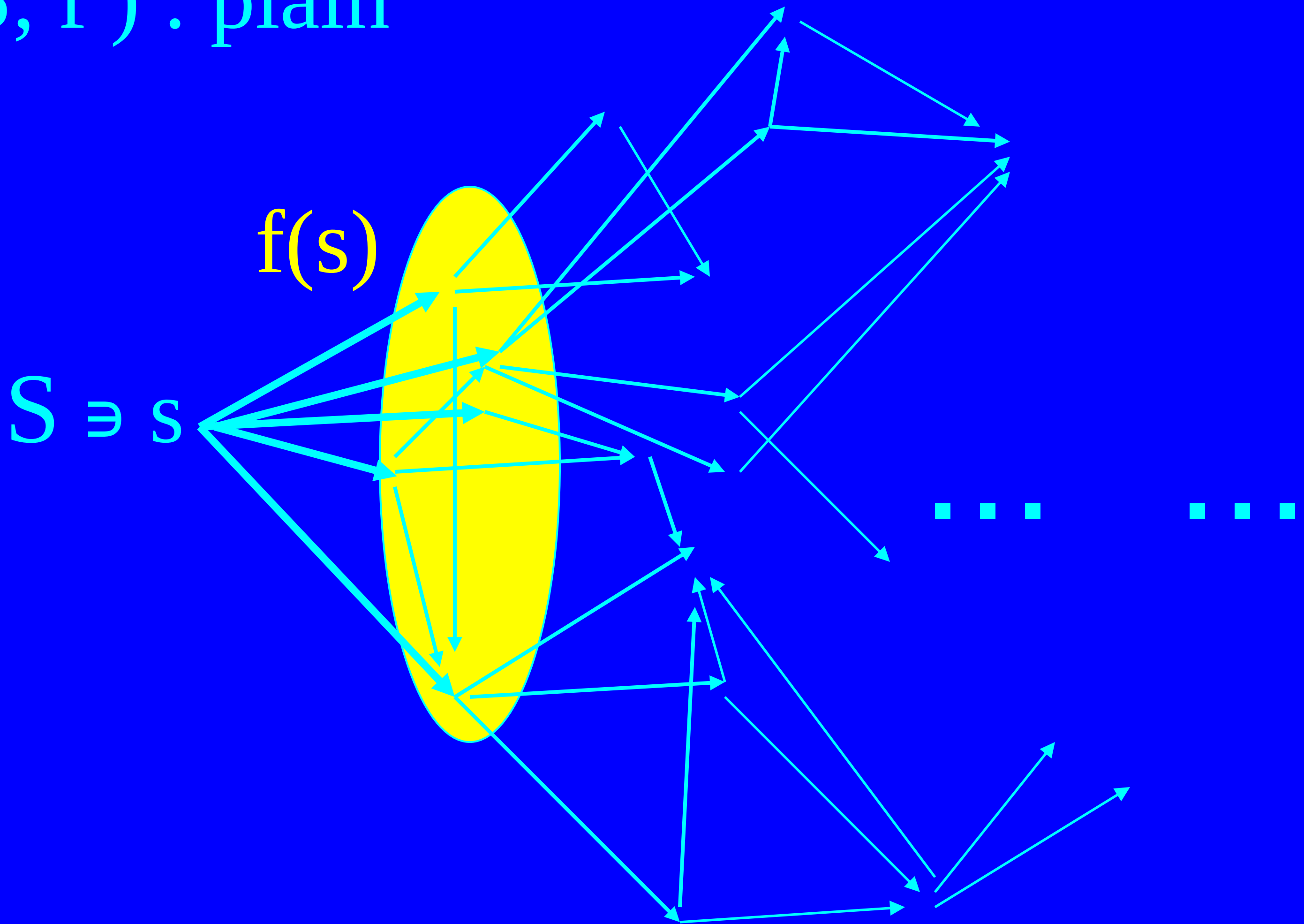


Plain algorithm (4)



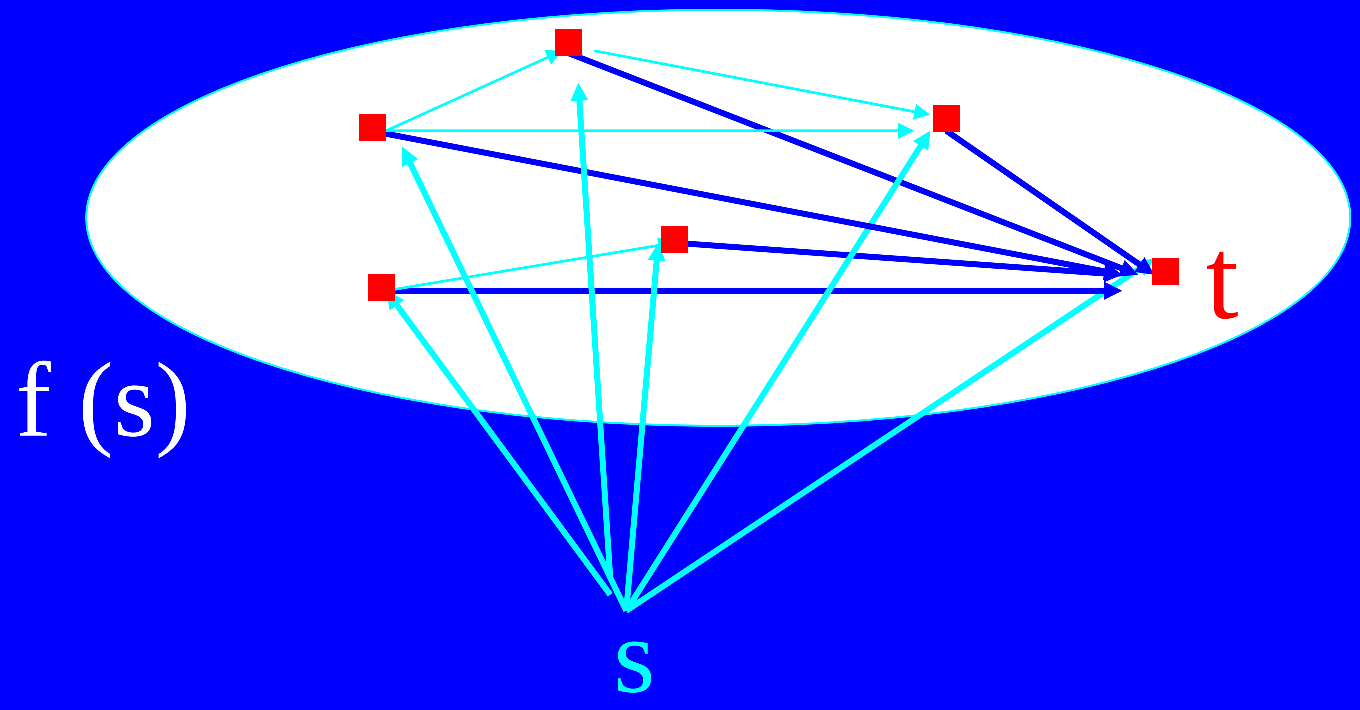
f (s) = diagram at s

(S, f) : plain

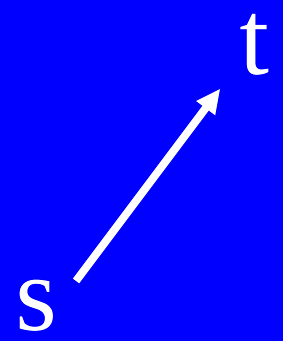


$$H_s(t) = \{f(s) \cap f^{-1}(t)\} \cup \{t\}$$

the hook of t



subtraction of a hook

If 

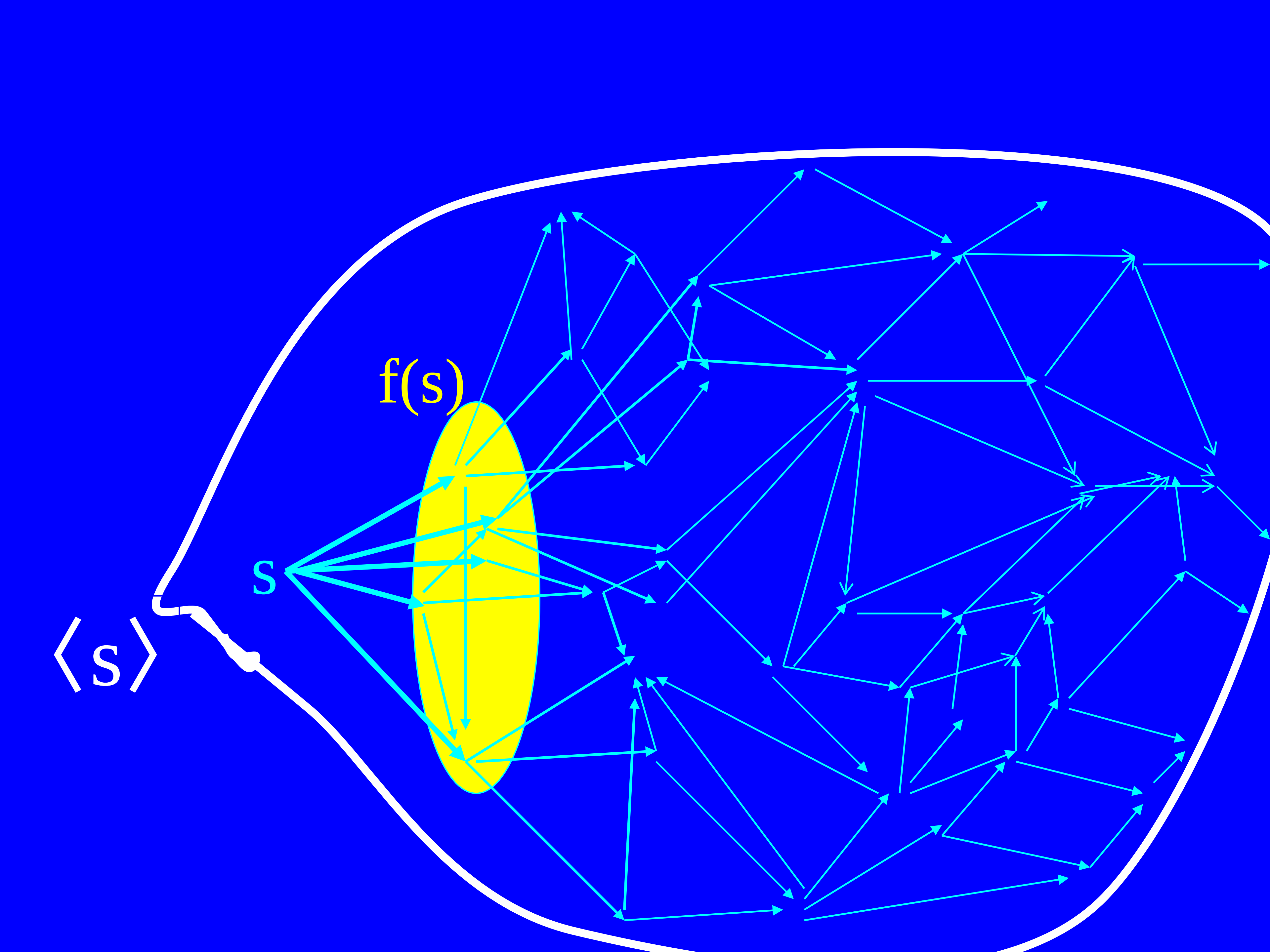
, then

$$f(t) \cong f(s) - H_s(t).$$

(Each arrow represents a hook subtraction process.)

Fundamental Theorem of Plain Algorithms

The diagram f(s) equipped with the hook structure knows everything about ⟨s⟩.


f(s)
s
⟨s⟩

In what follows, we assume:

(S, f) is a plain algorithm such that :

$$| f(s) | < \infty$$

for any $s \in S$.

Hook Formula (1)

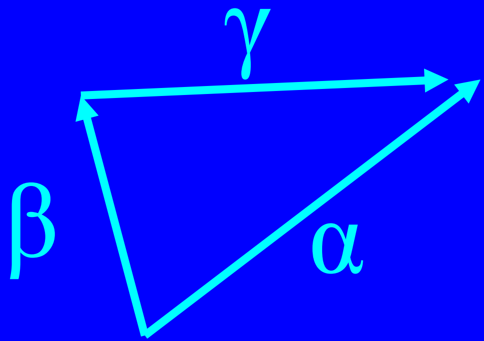
With an arrow $\alpha = (s \to t)$,

we associate an element

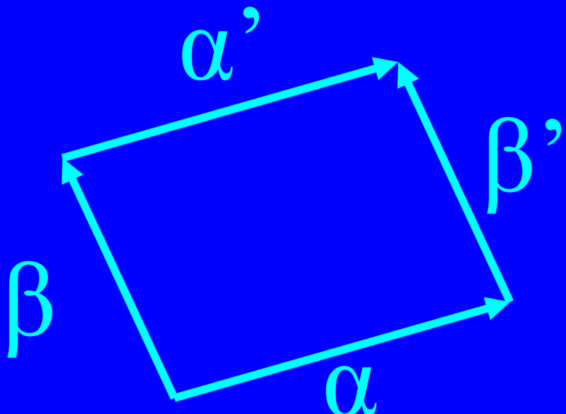
$F(\alpha)$ of a field K

in such a way that :

Hook Formula (2)



$$F(\alpha) = F(\beta) + F(\gamma)$$



$$F(\alpha) = F(\alpha')$$

$$F(\beta) = F(\beta')$$

Hook Formula (3)

With a path of finite length

$$p = (s \xrightarrow{\alpha} t \xrightarrow{\beta} u \xrightarrow{\gamma} \cdots),$$

we associate

$$F(p) = F(\alpha)+F(\beta)+F(\gamma)+ \cdots$$

($F(p) = 1$ if p is of length 0.)

Hook Formula (4)

Let $v \in \langle s \rangle$, $\langle s \setminus v \rangle = \{\langle s \rangle \setminus \langle v \rangle\} \cup \{v\}$,

$P(s \setminus v) = \{ s \to t \to \cdots \text{ in } \langle s \setminus v \rangle \}$

$$\Rightarrow \sum_{p \in P(s \setminus v)} F(p)^{-1} = \prod_{x \in f(s) \setminus f(v)} (1 + F(s \to x))^{-1}$$

: a skew version of Nakada's formula

(Osaka J.Math. 2008)

(generalized) Sato’s game

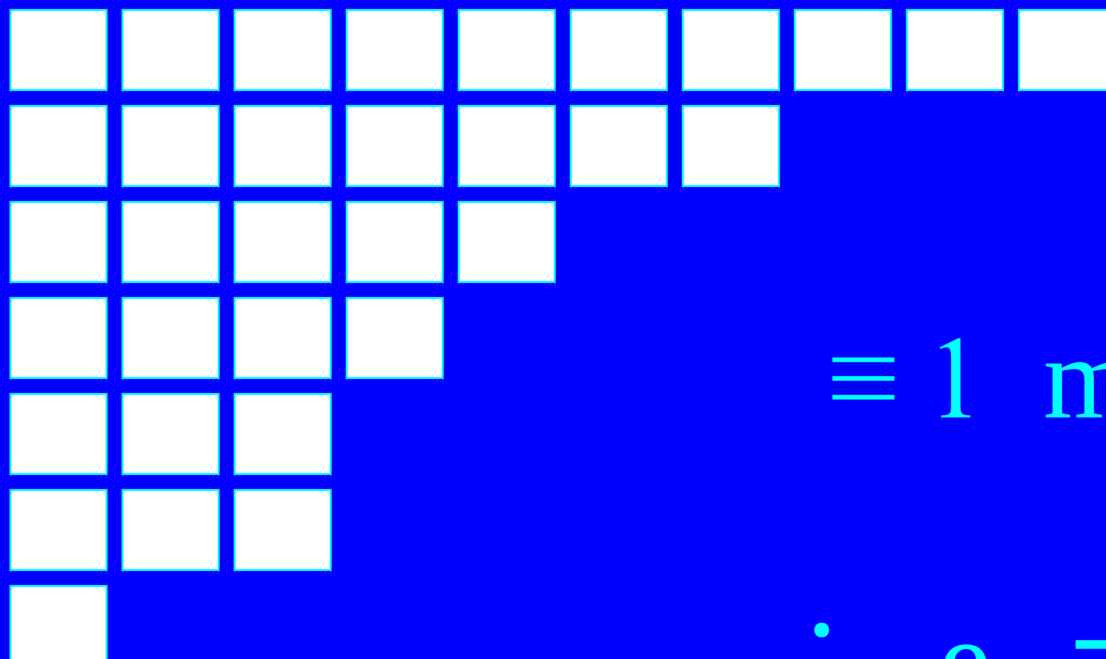
(Rule of the game)

Starting from a given diagram,
2 players alternatively subtract a hook.
A player who puroduces the empty
diagram Φ is a winner.

An equivalent game is known as Welter’s game.
(J.H. Conway : On Numbers and Games, Ch.13.)

2-adic expansion of Y (1)

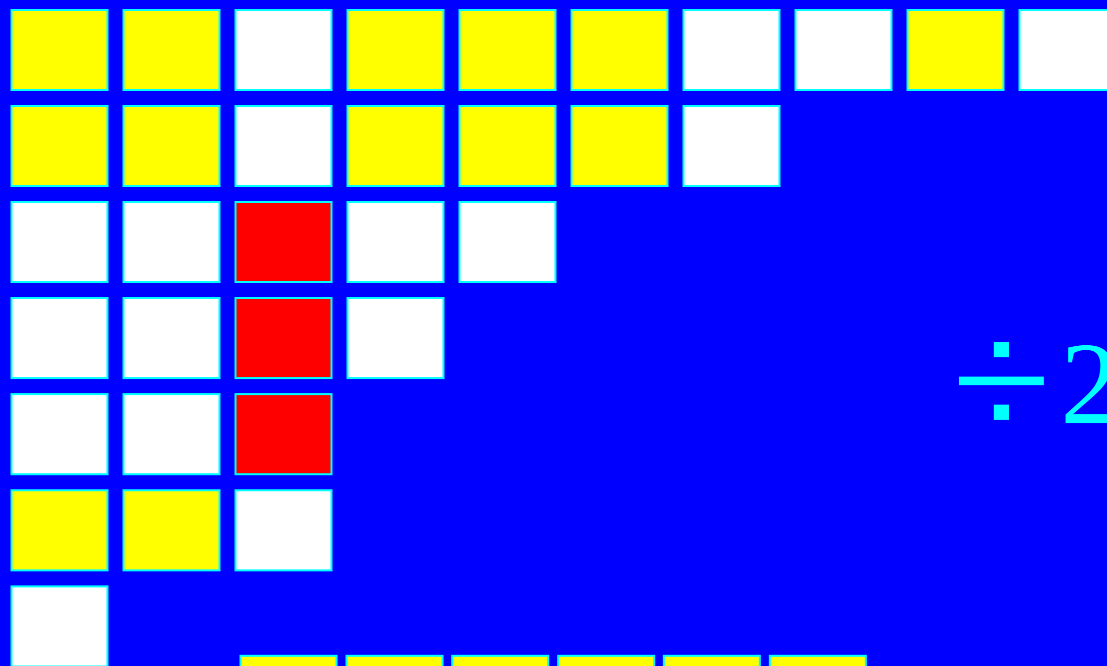
$Y =$



$\equiv 1 \mod 2$

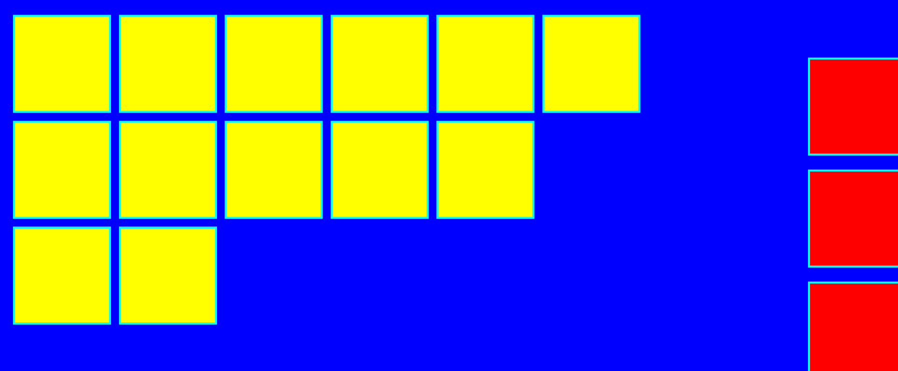
$\therefore\ e_0 = 1$

The quotient of



÷2

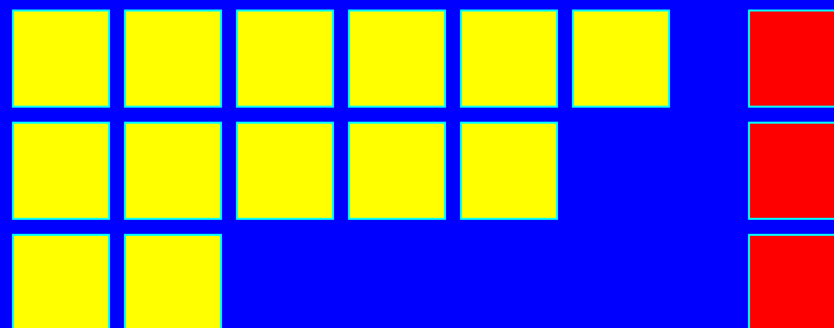
=



introduced by G. de Robinson

2-adic expansion of Y (2)

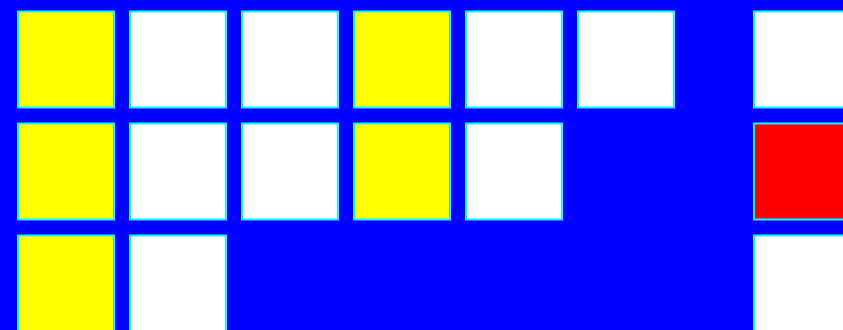
The quotient of $Y \div 2$ =

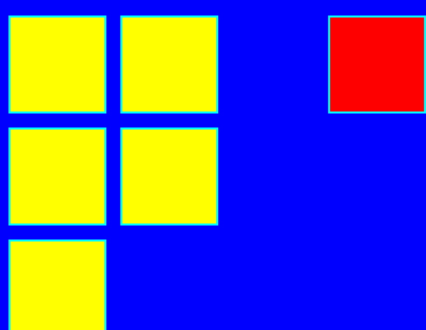


$\equiv 0 \mod 2$

$\therefore \; e_1 = 0$

2-adic expansion of Y (3)

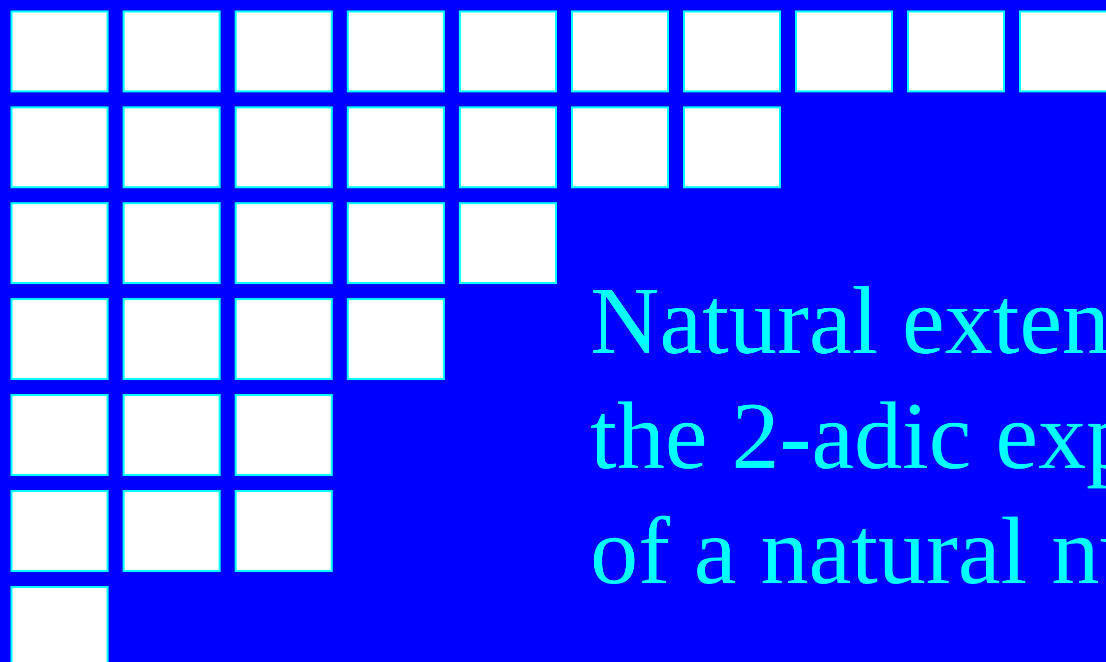
the quotient of $\left(\right.$  $\div 2 \left. \right)$

$=$  $\equiv 0 \mod 2$

$\therefore \quad e_2 = 0$

2-adic expansion of Y (4)

$Y =$



Natural extension of the 2-adic expansion of a natural number

$$E(Y) = \cdots e_4 e_3 e_2 e_1 e_0 = \cdots 010001$$

(generalized) Sato's game (2)

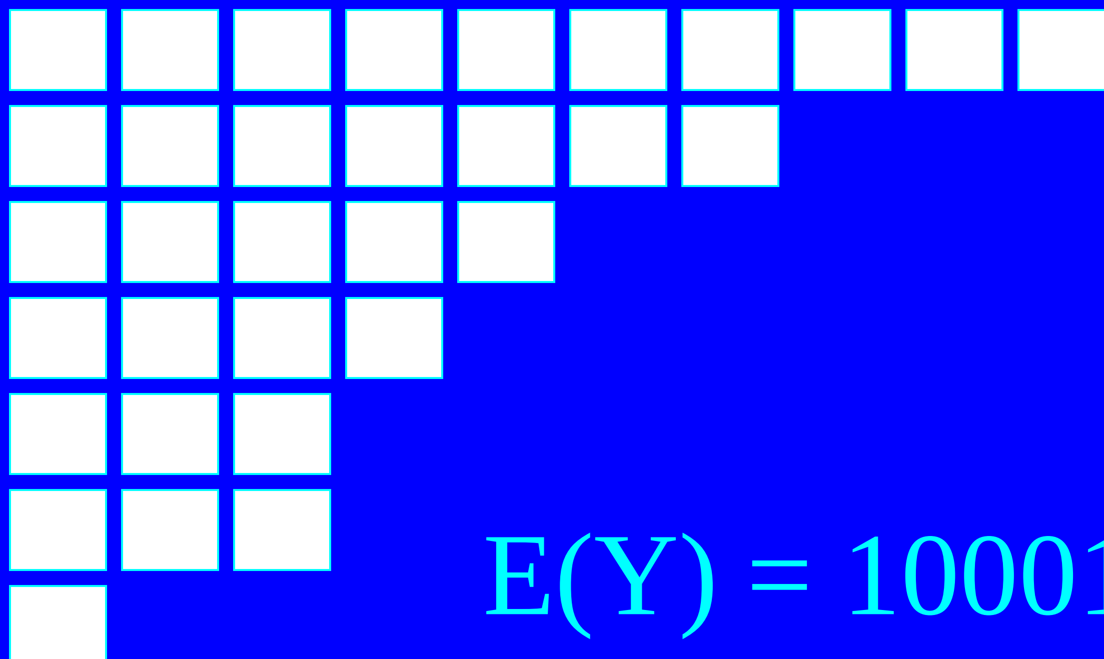
This game can be analyzed completely :

The diagram Y is a winning position for the 2nd player (resp. 1st player)

$\Longleftrightarrow$ $E(Y) = 0$ (resp. $E(Y) \neq 0$)

Find out good moves.

Y=



$E(Y) = 10001 \neq 0$

Klein's 4-group G

$G = \langle A, B \rangle \quad AB = BA$

$A^2 = B^2 = E$

$\langle AB \rangle = \{ E, AB \}$

$\langle A \rangle = \{ E, A \}$

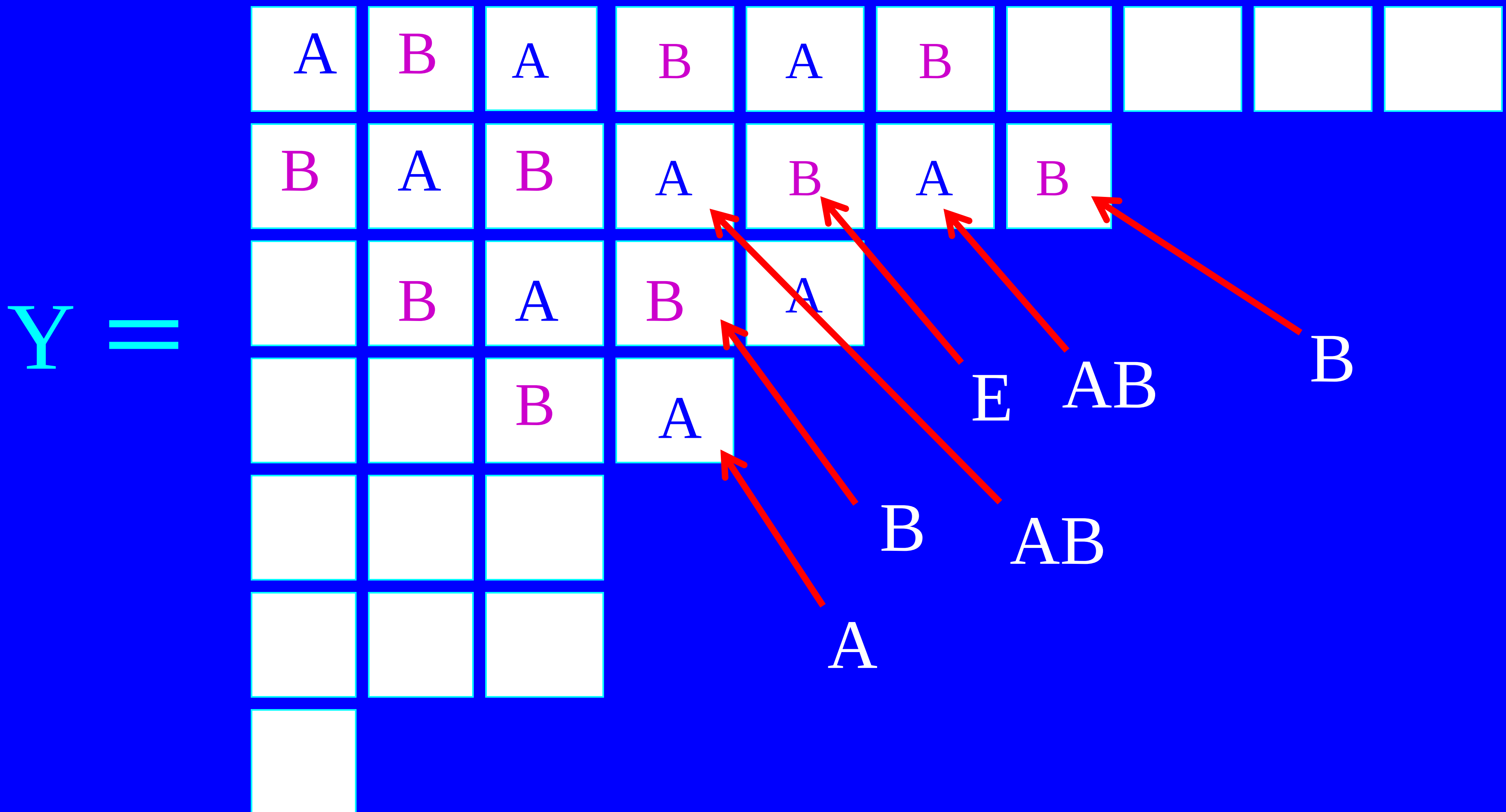
$\langle B \rangle = \{ E, B \}$

Put A & B alternatively into the boxes.

Y =

A	B	A	B	A	B				
B	A	B	A	B	A	B			
	B	A	B	A					
		B	A						

Multiply A & B's in a hook



Y =

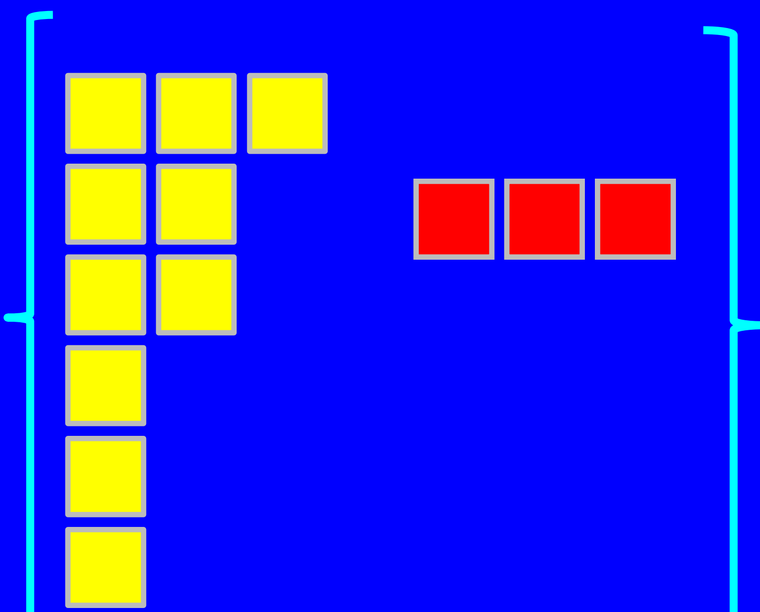
E	AB	B	AB	E	AB	B	A	AB	B
E	A	B	A	E	AB	B			
A	B	AB	B	A					
B	A	E	A						
A	B	AB							
E	AB	B							
A									

$\langle A \rangle$- quotient $Y_{\langle A \rangle}$

$Y =$

E	AB	B	AB	E	AB	B	A	AB	B
E	AB	B	AB	E	AB	B			
A	B	AB	B	A					
B	A	E	A						
A	B	AB							
E	AB	B							
A									

$Y_{\langle A \rangle} =$



$$E(Y) = \; 01\underline{\cdots}10\underline{\cdots}010\underline{\cdots}0$$

j 1's

i 0's

h 0's

⟨ B ⟩-quotient
j times

⟨ A ⟩-quotient
i times

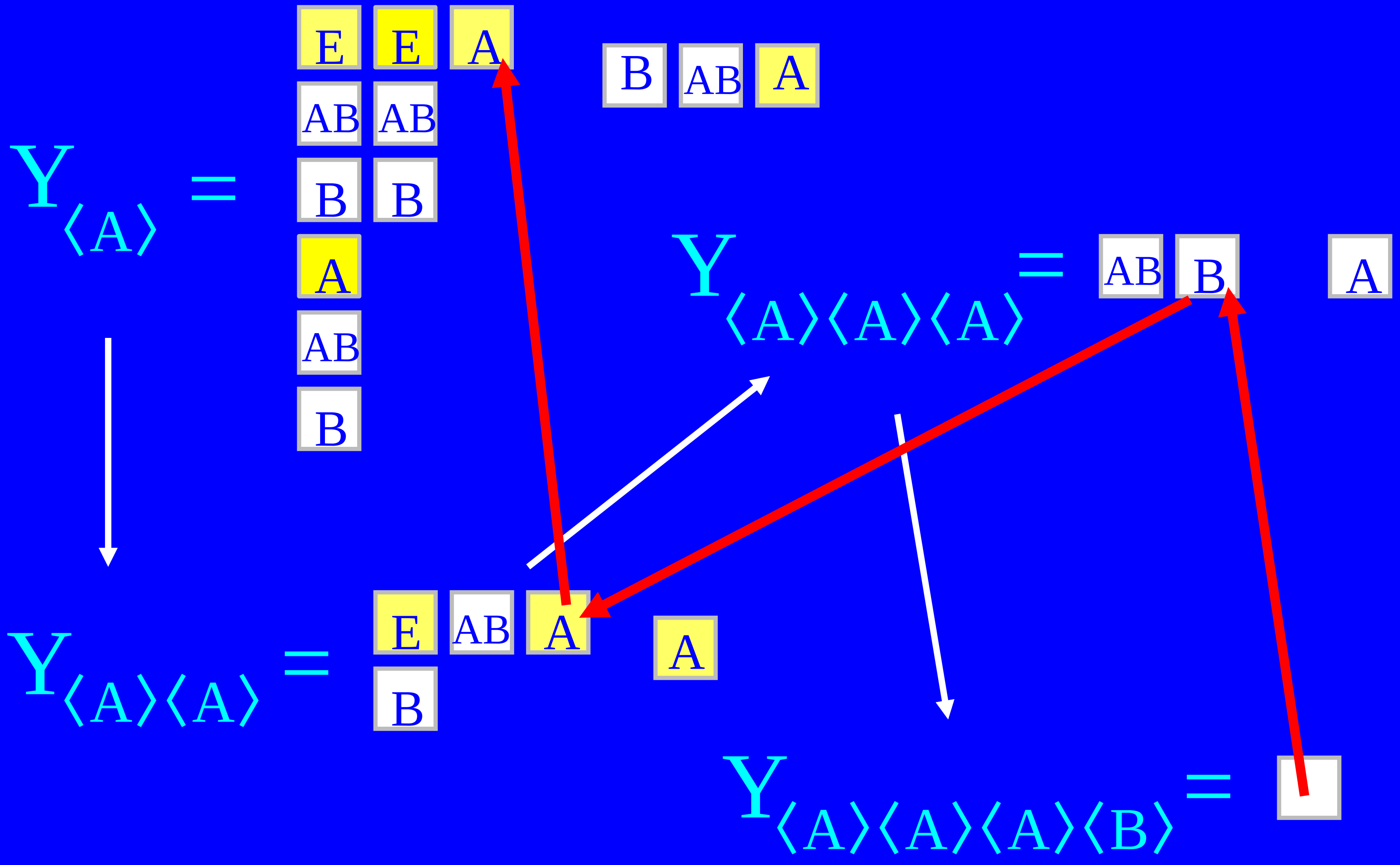
⟨ AB ⟩-quotient
h times

In the above example,

$$E(Y) = 10001$$

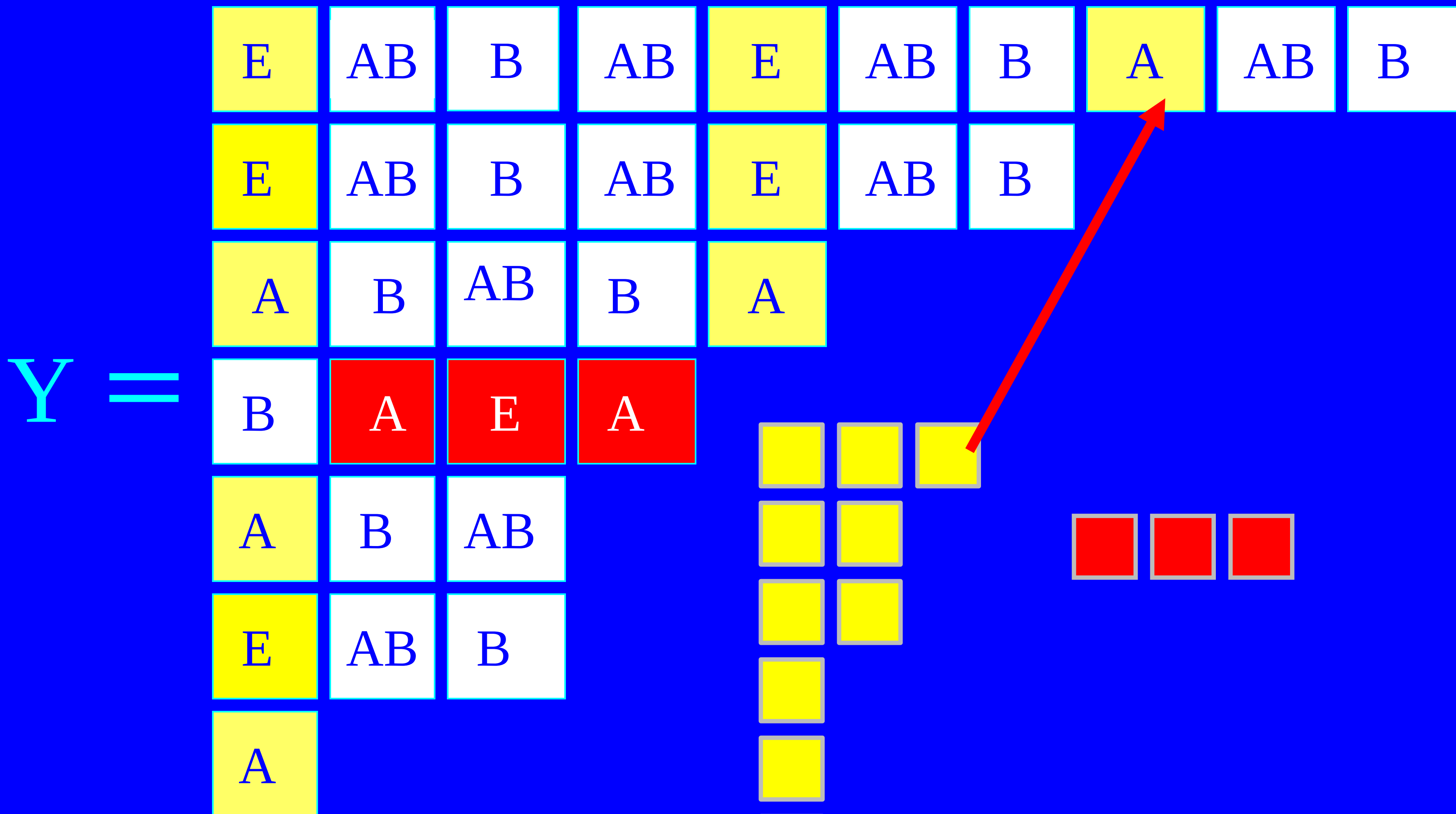
$\langle A \rangle$-quotient

$\langle B \rangle$-quotient



Y⟨A⟩ =
E E A
AB AB
B B
A
AB
B
B AB A
Y⟨A⟩⟨A⟩⟨A⟩ =
AB B A
Y⟨A⟩⟨A⟩ =
E AB A
B
A
Y⟨A⟩⟨A⟩⟨A⟩⟨B⟩ =

Subtracting the hook of this box is the unique good move.



Thank you.