Local and global analyticity for μ-Camassa-Holm equations

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1. Camassa-Holm equation and the μ-Camassa-Holm equation

Camassa-Holm equation $u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}$ or

$$u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1}\left[u^2 + \frac{1}{2}u_x^2\right] = 0.$$

Shallow water wave, bi-Hamiltonian structure, integrability,...

μ-Camassa-Holm equation (μCH)

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx},\ x \in S^1 = \mathbb{R}/\mathbb{Z},$$

$$\mu(\varphi) := \int_{S^1} \varphi(x)\, dx \ (\mu \text{ for mean value}).$$

Liquid crystal, group of diffeos of S^1,...
Set $A(\varphi) = \mu(\varphi) - \varphi_{xx}$, then μCH is

$$u_t + uu_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0.$$

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2. μ-equations

$$\mu(\varphi) := \int_{S^1} \varphi(x)\, dx, \quad A(\varphi) = \mu(\varphi) - \varphi_{xx}.$$

μ-Camassa-Holm equation (μCH) by Khesin-Lenells-Misiołek

$$u_t + uu_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0.$$

μ-Degasperis-Procesi equation (μDP) by Lenells-Misiołek

$$u_t + uu_x + \partial_x A^{-1}\left[3\mu(u)u\right] = 0.$$

Higher order μ-Camassa-Holm equation by Wang-Li-Qiao

$$u_t + uu_x + \partial_x B^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2 - 3u_x u_{xxx} - \frac{7}{2}u_{xx}^2\right] = 0,$$

$$B(\varphi) = \mu(\varphi) + (-\partial_x^2 + \partial_x^4)\varphi,$$

3. Pseudodifferential operators

$$(1-\partial_x^2)^{-1}\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\xi^2)^{-1}\hat{\varphi}(\xi)\,d\xi.$$

$A(\varphi) = \mu(\varphi) - \varphi_{xx}$, where $\mu(u) = \int_{S^1} u\,dx$.
For $\varphi = \sum_{k\in\mathbb{Z}} a_k e^{2k\pi ix}$, we have $\mu(\varphi) = a_0$ and

$$A(\varphi) = a_0 + \sum_{k\neq 0} 4\pi^2k^2a_ke^{2k\pi ix}, \quad \mathrm{ord} = 2$$

$$A^{-1}(\varphi) = a_0 + \sum_{k\neq 0} \frac{a_k}{4\pi^2k^2}e^{2k\pi ix} \quad \mathrm{ord} = -2.$$

The action is diagonal.
Some authors describe A^{-1} in terms of an integral kernel, but the series expression is simpler.

4. Formulation of IVPs

CH and μCH involves a pseudodifferential operators (Fourier multipliers)

$$A = \left[\mu(\cdot) - \partial_x^2\right]^{-1}.$$

So research must be GLOBAL in x

IVP

$$u_t + uu_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0,$$
$$u(0,x) = u_0(x)$$

can be solved LOCALLY or GLOBALLY in t.

CK (Ovsyannikov) type time-local argument is possible.

Solution in a suitable space of functions on S^1_x.

Things are global in x, while they can be either local or global in t.

5. IVP in Sobolev spaces: known results

Consider initial value problems for μCH, μDP, the higher-order μCH.
Let $u_0(x)$ be the initial value.
Local(-in-time) well-posedness and global existence in $H^s(S^1)$ (s is sufficiently large) has been established by Khesin-Lenells-Misiołek 2008, Lenells-Misiołek-Tiğray 2010, Wang-Li-Qiao 2018.

For global(-in-time) existence, we assume that $(\mu - \partial_x^2)u_0(x)$ does not change signs in the cases of μCH and μDP.

This is inspired by the result about the original CH: assumption is that $(1 - \partial_x^2)u_0(x)$ (the McKean quantity) does not change signs.

6. Local and global analyticity

THE GOAL OF THIS TALK IS:

IVP for μCH, μDP, the higher-order μCH with analytic initial value (with some technical assumptions) on S^1.

$\Rightarrow$ Unique existence of global-in-time analytic solution

Ref: (generalized) CH, Barostichi-Himonas-Petronilho 2017

WHAT REMAINS TO BE PROVED (solvability in H^s is known):

1. analyticity in x ($t > 0$ fixed)$\leftarrow$ Kato-Masuda theory
2. analyticity in t and x, local in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument
3. global analyticity in t
4. From '$t \mapsto u(t, \cdot)$ analytic' (mapping from $\mathbb{R}_t$ to a function space in x) to 'analytic in (t, x)' (mapping from $\mathbb{R}^2_{t,x}$ to $\mathbb{C}$)

CK argument alone is not good enough (next slide).

7. CK argument alone is not good enough.

We want to prove global-in-time analyticitiy.
Cauchy-Kowalevsky is local.
We employ the known global-in-time solvability in H^s. How?

Let T^* be the sup of T such that u is analytic in t up to $t = T$.
We want to prove $T^* = \infty$ by contradiction.
Assume otherwise, i.e. $T^* < \infty$.

CK does not guarantee the well-definedness of $u(T^*)$,
but the H^s solvability implies the existence of $u(T^*) \in H^s$, which is analytic in x by the Kato-Masuda theory.
We apply the CK at $t = T^*$ and extend the lifespan.
It contradicts the assumption $T^* =$ 'sup of lifespan'.

8. Analytic Sobolev spaces on S^1

For a function φ on $S^1 = \mathbb{R}/\mathbb{Z}$, we set $\hat{\varphi}(k) = \int_{S^1} \varphi(x) e^{-2k\pi i x}\, dx$.
Following Barostichi-Himonas-Petronilho 2015, we introduce

$$G^{\delta,s} = \left\{ \varphi \in L^2(S^1);\, \|\varphi\|_{\delta,s} < \infty \right\},\ \|\varphi\|_{\delta,s}^2 = \sum_{k\in\mathbb{Z}} \langle k \rangle^{2s} e^{2\delta|k|} |\hat{\varphi}(k)|^2.$$

$G^{\delta,s} \hookrightarrow G^{\delta',s}$ and $G^{\delta,s} \hookrightarrow G^{\delta,s'}$ if $0 < \delta' < \delta \leq 1, 0 < s' < s$.
$\left\{ G^{\delta,s} \right\}_{0<\delta\leq 1}$ is a (decreasing) scale of Banach spaces.

If $\varphi \in G^{\delta,s}$, then this function on $\mathbb{R}/\mathbb{Z} = S^1$ has an analytic extension to $\{x + iy \in \mathbb{C};\, |y| < \delta/(2\pi)\}$.

Any analytic function on S^1 belongs to $G^{\delta,s}$ for some δ and any s.

If $s > 1/2$,

$$\|\varphi\psi\|_{\delta,s} \leq c_s \|\varphi\|_{\delta,s} \|\psi\|_{\delta,s}, \quad c_s = \left[2(1 + s^{2s}) \sum_{k=0}^{\infty} \langle k \rangle^{-2s} \right]^{1/2}.$$

9. Pseudodifferential operators A and A^{-1}

$A(\varphi) = \mu(\varphi) - \varphi_{xx}$, where $\mu(u) = \int_{S^1} u\,dx$.
For $\varphi = \sum_{k\in\mathbb{Z}} a_k e^{2k\pi ix}$, we have $\mu(\varphi) = a_0$ and

$$A(\varphi) = a_0 + \sum_{k\neq 0} 4\pi^2 k^2 a_k e^{2k\pi ix}, \quad \mathrm{ord} = 2$$

$$A^{-1}(\varphi) = a_0 + \sum_{k\neq 0} \frac{a_k}{4\pi^2 k^2} e^{2k\pi ix} \quad \mathrm{ord} = -2.$$

$A\colon G^{\delta,s+2} \to G^{\delta,s}$ and $A^{-1}\colon G^{\delta,s} \to G^{\delta,s+2}$ bounded.

μCH

$$u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0.$$

First-order ΨDE of the ‘Kowalevski type’.

10. Local analytic CP for μCH

Theorem (Y.)

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

Let $s > 1/2$.
If $u_0 \in G^{\Delta,s+1}$, then there exists a positive time $T = T(u_0, s, \Delta)$ such that for every $d \in (0,1)$,
the Cauchy problem (1) *has a unique solution in t which is a holomorphic function valued in $G^{\Delta d,s+1}$ in the disk $|t| \leq T(1-d)$.*

Ref: CH and similar equations, Barostichi-Himonas-Petronilho 2015

11. Local analytic CP for μCH: proof

Assume $\Delta = 1$ for simplicity. Assume $0 < \delta' < \delta \leq 1$.

$$F(u) := -\frac{1}{2}(u^2)_x - \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right]$$

We want to show that F satisfies the Lipschitz-Cauchy estimate:

$$\|F(u) - F(v)\|_{\delta',s+1} \leq \frac{\text{const.}}{\delta - \delta'}\|u - v\|_{\delta,s+1}.$$

This estimate follows from

$$\|\varphi_x\|_{\delta',s} \leq \frac{2\pi e^{-1}}{\delta - \delta'}\|\varphi_x\|_{\delta,s}$$

$$\|\varphi_x\|_{\delta,s} \leq 2\pi\|\varphi_x\|_{\delta,s+1}$$

$$\|\partial_x A^{-1}(\varphi)\|_{\delta',s+1} \leq \frac{2\pi e^{-1}}{\delta - \delta'}\|\varphi\|_{\delta,s}$$

and some algebra like $u_x^2 - v_x^2 = (u+v)_x(u-v)_x$,
$\mu(u)u - \mu(v)v = \mu(u)(u-v) + [\mu(u) - \mu(v)]v$.

12. Global theory: global solvability in H^s

Theorem (Khesin-Lenells-Misiołek)

Let $s > 5/2$. Assume that $u_0 \in H^s(S^1)$ has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \text{ (or } \leq 0).$$

Then the Cauchy problem for the μCH equation

$$\begin{cases} u_t + uu_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0, \\ u(0,x) = u_0(x) \end{cases}$$

has a unique solution in $\mathcal{C}(\mathbb{R}_t, H^s(S^1_x)) \cap \mathcal{C}^1(\mathbb{R}_t, H^{s-1}(S^1_x))$. Moreover, local well-posedness (in particular, uniqueness) holds.

13. Main result

Theorem (Y.)

Assume that a real-analytic function u_0 on S^1 has non-zero mean and satisfies the condition

$$(\mu - \partial_x^2)u_0 \geq 0 \ (\text{or} \leq 0).$$

Then the Cauchy problem

$$\begin{cases} u_t + uu_x + \partial_x A^{-1}\left[2\mu(u)u + \frac{1}{2}u_x^2\right] = 0, \\ u(0,x) = u_0(x) \end{cases}$$

has a unique solution $u \in \mathcal{C}^\omega(\mathbb{R}_t \times S_x^1)$.

Similar statements hold true for μDP and the higher order μCH. In the higher order case, u_0 can be an arbitrary analytic function.

14. Radius of analyticity

$$S(\delta) = \{z = x + iy \in \mathbb{C};\, |y| < \delta)\},$$
$$A(\delta) = \Big\{f \colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R};\, f \text{ has an analytic continuation to } S(\delta)\Big\}.$$

In the previous theorem, let $u_0 \in A(r_0)$. Fix $\sigma_0 < (\log r_0)/(2\pi)$ and set

$$K = (20\pi d_1 + 8 + 4\pi^2 c_1)\,[1 + \max\{\|u(t)\|_2;\, t \in [-T, T]\}],$$
$$\sigma(t) = \sigma_0 - \frac{\sqrt{6}\pi\gamma}{K}\|u_0\|_{(\sigma_0,2)}(e^{K|t|/2} - 1).$$

Then, for any fixed $T > 0$, we have $u(\cdot, t) \in A(e^{2\pi\sigma(t)})$ for $t \in [-T, T]$.

15. Regularity theorem by Tosio Kato and Kyûya Masuda 1

Consider the equation

$$\frac{du}{dt} = F(u),\ u(0) = u_0.$$

Here F is typically a (nonlinear) continuous mapping from Banach space to another, possibly involving ΨDOs in x.
Kato-Masuda theorem gives some sufficient condition for the regularity of $u(t), t > 0$.
If u_0 is regular to some extent, then so is $u(t), t > 0$.

Let $\{\Phi_s; -\infty < s < \infty\}$ be a family of functions on a Banach space, typically $\frac{1}{2}\|\cdot\|_s^2$, where $\{\|\cdot\|_s\}_s$ is a family of norms.

If $\Phi_s(u)$ is bounded, then u is regular in some sense.
So we want bounds on $\Phi_s(u)$ in terms of u_0.

16. Regularity theorem by Kato-Masuda 2

X, Z: Banach spaces and Z is a dense subspace of X.
F : continuous mapping from Z to X.
$\mathcal{O}$: an open subset of Z.
$\{\Phi_s; -\infty < s < \infty\}$: a family of real-valued functions on Z.
Assume there exist positive constants K and L satisfying

$$|\langle F(v), D\Phi_s(v)\rangle| \leq K\Phi_s(v) + L\Phi_s(v)^{1/2}\partial_s\Phi_s(v),\ v \in \mathcal{O}.$$

D: Frechét derivative
$\langle\cdot,\cdot\rangle$ (no subscript) : the pairing of X and $\mathcal{L}(X;\mathbb{R})$.

If $du/dt = F(u),\ u(0,x) = u_0(x)$, then for a fixed constant s_0 there exists a function $s(t)$ such that

$$\Phi_{s(t)}(u(t)) \leq \Phi_{s_0}(u_0)e^{Kt},\ t \in [0,T].$$

If u_0 is regular to some extent, then so is $u(t), t > 0$.

17. Regularity theorem by Kato-Masuda: summary

Φ_s is essentially a norm. $\Phi_s(\cdot) = \|\cdot\|_s^2/2$ for some $\|\cdot\|_s$.
A bound in terms of Φ_s is a measure of regularity.

If $|\langle F(v), D\Phi_s(v)\rangle| \leq K\Phi_s(v) + L\sqrt{\Phi_s(v)}\,\partial_s\Phi_s(v)$,
then the solution to

$$\frac{du}{dt} = F(u),\ u(0) = u_0$$

satisfies $\Phi_{s(t)}(u(t)) \leq \Phi_{s_0}(u_0)e^{Kt}$, $t \in [0, T]$ for some function $s(t)$.

If u_0 is regular to some extent, then so is $u(t), t > 0$.

Ref: Kato-Masuda, 1986 (KdV and similar equations, analyticity in x only, not in t.)

18. Proof of analyticity

μCH obviously has the form $du/dt = F(u)$.

We have to choose SUITABLE FUNCTION SPACES and Φ_s and then prove

$$|\langle F(v), D\Phi_s(v)\rangle| \leq K\Phi_s(v) + L\sqrt{\Phi_s(v)}\,\partial_s\Phi_s(v).$$

$A(\delta) = \Big\{ f\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R};\ f$ has an analytic continuation to $|y| < \delta \Big\}$.

$A(\delta)$ is Fréchet . The (semi)norms $\|v\|^2_{(\sigma,2)} = \sum_{j=0}^{\infty} \frac{1}{j!^2} e^{4\pi\sigma j} \|v^{(j)}\|_2^2$.

Set $\Phi_{\sigma,m}(v) = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \frac{\|v^{(j)}\|_2^2}{2} (= \Phi_s)$

Kato-Masuda $\Rightarrow$ Bounds on $\Phi_{\sigma,m}(u(t))$ and $\|u(t)\|^2_{(\sigma,2)}$.

$G^{\delta,s}$ and $A(\delta)$ can be used interchangeably.

$G^{\delta,s} \subset A(\delta)$ and $A(\delta) \subset G^{\delta',s} (\delta' < \delta)$.

19. Estimates

Set $X = H^{m+2}, Z = H^{m+5}$,

$$\Phi_{\sigma,m}(v) = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \frac{\|v^{(j)}\|_2^2}{2}$$

$$F_\mu(u) = -uu_x - \partial_x A^{-1} \left[2\mu(u)u + \frac{1}{2}u_x^2 \right].$$

Then, for $v \in H^{m+5}$, we have

$$|\langle F_\mu(v), D\Phi_{\sigma,m}(v)\rangle|$$
$$\leq \text{const.}\|v\|_2 \Phi_{\sigma,m}(v) + \text{const.}\Phi_{\sigma,m}(v)^{1/2} \partial_\sigma \Phi_{\sigma,m}(v).$$

Kato-Masuda theory works. The solution to μCH IVP satisfies a bound in terms of $\Phi_{\sigma,m}$.
$\|u(t)\|^2_{(\sigma,2)} < \infty$ for some σ and $u(t)$ is analytic in x for any $t > 0$.

20. Estimating $\langle F_\mu(v), D\Phi_{\sigma,m}(v)\rangle$

$\langle\cdot,\cdot\rangle$ is the pairing of H^{m+2} and $(H^{m+2})^* \simeq H^{m+2}$,
$\langle\cdot,\cdot\rangle_2$ is the H^2 inner product.

$$\langle F_\mu(v), D\Phi_{\sigma,m}(v)\rangle = \sum_{j=0}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} \underline{\langle v^{(j)}, \partial_x^j F_\mu(v)\rangle_2},$$

$$\underline{\langle v^{(j)}, \partial_x^j F_\mu(v)\rangle_2} = -\langle v^{(j)}, \partial_x^j (v v_x)\rangle_2 - 2\langle v^{(j)}, \partial_x^{j+1} A^{-1}[\mu(v)v]\rangle_2 - \frac{1}{2}\langle v^{(j)}, \partial_x^{j+1} A^{-1}(v_x^2)\rangle_2.$$

Estimates by using

- $A^{-1}\colon H^{s+2} \to H^2$ bounded.
- $\|\varphi\psi\|_0 \le \mathrm{const.}\|\varphi\|_0\|\psi\|_s$, where $\|\cdot\|_s$ is the H^s norm.
- $\|\varphi\psi\|_2 \le \gamma(\|\varphi\|_1\|\psi\|_2 + \|\varphi\|_2\|\psi\|_1)$ for some constant $\gamma > 0$
  (a variant of the Kato-Ponce inequality)
  in particular, H^2 is closed under multiplication.
  (The original Kato-Ponce is about the $W^{m,p}$ and L^∞ norms.)

21. Estimating $\sum_{j=0}^{m} j!^{-2} e^{4\pi\sigma j} \langle v^{(j)}, \partial_x^j (v v_x) \rangle_2$

We encounter (Leibnitz rule, $\ell = 0$ is dealt with separately)

$$Q_j = \sum_{\ell=1}^{j} \binom{j}{\ell} \langle v^{(j)}, v^{(\ell)} v^{(j-\ell+1)} \rangle_2. \quad \text{(degree 3)}$$

Apply Schwarz and get $\|v^{(j)}\|_2 \|v^{(\ell)} v^{(j-\ell+1)}\|_2$. By Kato-Ponce,

$$\begin{aligned}\|v^{(\ell)} v^{(j-\ell+1)}\|_2 &\le \gamma \left(\|v^{(\ell)}\|_2 \|v^{(j-\ell+1)}\|_1 + \|v^{(\ell)}\|_1 \|v^{(j-\ell+1)}\|_2 \right) \\ &\le 2\pi\gamma \left(\|v^{(\ell)}\|_2 \|v^{(j-\ell)}\|_2 + \|v^{(\ell-1)}\|_2 \|v^{(j-\ell+1)}\|_2 \right).\end{aligned}$$

Combining this estimate with the Schwarz inequality, we get

$$\begin{aligned}|Q_j| &\le 2\pi\gamma (Q_{j,1} + Q_{j,2}), \\ Q_{j,1} &= \|v^{(j)}\|_2 \sum_{\ell=1}^{j} \binom{j}{\ell} \|v^{(\ell)}\|_2 \|v^{(j-\ell)}\|_2, \\ Q_{j,2} &= \|v^{(j)}\|_2 \sum_{\ell=1}^{j} \binom{j}{\ell} \|v^{(\ell-1)}\|_2 \|v^{(j-\ell+1)}\|_2.\end{aligned}$$

Set $b_k = k!^{-1} e^{2\pi\sigma k} \|v^{(k)}\|_2 \, (k = 0, 1, \ldots, j)$. Then we have

$$\frac{1}{j!^2} e^{4\pi\sigma j} Q_{j,1} \le \sum_{\ell=1}^{j} b_j b_\ell b_{j-\ell}. \quad \text{(degree 3)}$$

Set $B = \left(\sum_{j=0}^{m} b_j^2\right)^{1/2}, \widetilde{B} = \left(\sum_{j=1}^{m} j b_j^2\right)^{1/2}$.
$B^2 = \|v\|^2_{(\sigma,2,m)} = 2\Phi_{\sigma,m}(v)$, $\widetilde{B}^2 = (2\pi)^{-1} \partial_\sigma \Phi_{\sigma,m}(v)$.
Repeated use of the Schwarz inequality gives

$$\sum_{j=1}^{m} \frac{1}{j!^2} e^{4\pi\sigma j} Q_{j,1} \le \sum_{j=1}^{m} \sum_{\ell=1}^{j} b_j b_\ell b_{j-\ell} \le \sum_{\ell=1}^{m} \frac{b_\ell}{\sqrt{\ell}} \sum_{j=\ell}^{m} \sqrt{j} b_j b_{j-\ell}$$

$$\le B\widetilde{B} \sum_{\ell=1}^{m} \frac{\sqrt{\ell} b_\ell}{\ell} \le B\widetilde{B}^2 \left(\sum_{\ell=1}^{m} \frac{1}{\ell^2}\right)^{1/2} \le \frac{\pi}{\sqrt{6}} B\widetilde{B}^2$$

$$= \frac{1}{2\sqrt{3}} \sqrt{\Phi_{\sigma,m}(v)} \, \partial_\sigma \Phi_{\sigma,m}(v).$$

23. Final part of the proof of the Main result

1. Analyticity in x ($t > 0$ fixed)$\leftarrow$ Kato-Masuda, just completed
2. Analyticity in t and x, local in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument
3. Global analyticity in t $\leftarrow$ combination of 1 and 2
   Lifespan$< \infty$ leads to contradiction.
4. From '$t \mapsto u(t, \cdot)$ analytic' (mapping from $\mathbb{R}_t$ to a function space in x) to 'analytic in (t, x)' (mapping from $\mathbb{R}^2_{t,x}$ to $\mathbb{C}$)

The 4th step.

$\forall T > 0$, $t \in [-T, T] \mapsto u(t, \cdot) \in A(\delta_T)$ is analytic for some δ_T.

$$\left\| \partial_x^k \partial_t^j u \right\|_{L^2(S^1 \times [-T,T])} \leq \sqrt{2T} C^{j+k+1} (j+k)!.$$

We get a bound on $\left\| \Delta^\ell u \right\|_{L^2(S^1 \times [-T,T])}$ and u is analytic in (t, x) by Komatsu (1960) or Kotake-Narashimhan (1961).

Thank you very much!

Next year, I will talk about the Camassa-Holm system

$$\begin{cases} u_t + \beta u u_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0, \end{cases}$$

introduced by R. M. Chen and Y. Liu (2011).