Local and global analyticity for a generalized Camassa-Holm system

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1. (single) Camassa-Holm equation

Camassa-Holm equation $u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}$ or

$$u_t + uu_x + \partial_x(1-\partial_x^2)^{-1}\left[u^2 + \frac{1}{2}u_x^2\right] = 0 \text{ on } \mathbb{R},$$

where

$$(1-\partial_x^2)^{-1}\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\xi^2)^{-1}\hat{\varphi}(\xi)\,d\xi.$$

Shallow water wave, bi-Hamiltonian structure, integrability,...

variations:
periodic ($x \in S^1$),
μ (involves mean value on $S^1 \ni x$), Khesin-Lenells-Misiolek.
system

2. CH system of R. M. Chen-Y. Liu

Chen-Liu (IMRN 2011)

$$\begin{cases} u_t - u_{txx} - \alpha u_x + 3uu_x - \beta(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \qquad (1)$$

Here it is assumed that $u \to 0$ and $\rho \to 1$ hold as $|x| \to \infty$.
Set $v = \rho - 1 \to 0$.
(1) is equivalent to

$$\begin{cases} u_t + \beta uu_x + (1 - \partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0. \end{cases} \qquad (2)$$

with $u \to 0$, $v \to 0$ as $|x| \to \infty$.

3. Formulation of IVPs

The CH system of Chen-Liu

$$\begin{cases} u_t + \beta u u_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0. \end{cases}$$

with $u \to 0$, $v \to 0$ involves the ΨDO $(1-\partial_x^2)^{-1}$.
So research must be GLOBAL in x.

It can be solved LOCALLY or GLOBALLY in t.
Solutions in a suitable space of functions on $\mathbb{R}_x$.

4. Known result: time-global solvability in H^s

Theorem (Chen-Liu 2011)

Assume $0 < \beta < 2$, $s > 3/2$. *If* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *and* $\inf_{x\in\mathbb{R}} v_0(x) > -1$, *then the IVP for*

$$\begin{cases} u_t + \beta u u_x + (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2} u^2 + \frac{\beta}{2} u_x^2 + v + \frac{1}{2} v^2 \right] = 0, \\ v_t + u_x + (uv)_x = 0 \end{cases}$$

with $u(0,x) = u_0$, $v(0,x) = v_0$ *has a unique solution* (u,v) *in* $\mathcal{C}([0,\infty), H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})) \cap \mathcal{C}^1([0,\infty), H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$.

5. Main result: global analytic solution

If the initial data are analytic, then the solution is analytic globally in both t and x.

(μ-case is by Y., DCDS 2020)

For $r > 0$, set $S(r) = \{x + iy \in \mathbb{C};\ |y| < r\}$ and

$$A(r) = \{f : \mathbb{R} \to \mathbb{R};\ f(z) \text{ can be analytically continued to } S(r)\} \cap \left\{ f \in L^2_{x,y}(S(r')) \text{ for all } 0 < r' < r \right\}.$$

Theorem (**Global analyticity**)

Assume $0 < \beta < 2$ *and* $\inf_{x \in \mathbb{R}} v_0(x) > -1$.
If $u_0, v_0 \in A(r_0)$ *for some* $r_0 > 0$*,*

then the solution (u, v) *is analytic in* t*,x. It belongs to*

$\oplus^2 \mathcal{C}^\omega([0,\infty)_t \times \mathbb{R}_x)$.

6. time-local and global analyticity

IVP for the CH system with analytic initial value (with some technical assumptions).
$\Rightarrow$ Unique existence of a global-in-time analytic solution

Ref: (generalized) CH, Barostichi-Himonas-Petronilho 2017

WHAT REMAINS TO BE PROVED (solvability in H^s is known):

1. local analyticity in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument
2. analyticity in x ($t > 0$ fixed)
   $\leftarrow$ Kato-Masuda theory. The most difficult part.
3. global analyticity in t

7. $A(r)$ (Fréchet) and $E_{\delta,s}$ (Banach)

Following BHP (with some generalization and a modified notation), we introduce

$$\|f\|_{(\delta,s)} = \sup_{k\geq 0} \frac{\delta^k (k+1)^2 \|f^{(k)}\|_s}{k!} \; (0 < \delta \leq 1, s \geq 2).$$

and the Banach space $E_{\delta,s}$ by

$$E_{\delta,s} = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}); \|f\|_{(\delta,s)} < \infty \right\}.$$

$E_{\delta,s}$ is closed under multiplication.
$E_{\delta,s}$ is continuously embedded in $A(\delta)$.
Conversely, if $\delta < r/e$ then $A(r)$ is continuously embedded in $E_{\delta,s}$.

8. Continuity of operations on $E_{\delta,s}$

If $0 < \delta \leq 1$, $s \geq 2$, then

$$\|uv\|_{(\delta,s)} \leq \text{const.}\|u\|_{(\delta,s)}\|v\|_{(\delta,s)}.$$

If $0 < \delta' < \delta \leq 1$, we have

$$\|\partial_x u\|_{(\delta',s)} \leq \frac{1}{\delta - \delta'}\|u\|_{(\delta,s)},$$

$$\|\partial_x u\|_{(\delta,s)} \leq \|u\|_{(\delta,s+1)},$$

$$\|(1-\partial_x^2)^{-1}\partial_x^p u\|_{(\delta,s)} \leq \|u\|_{(\delta,s)} \; (p = 0,1,2),$$

$$\|(1-\partial_x^2)^{-1}\partial_x u\|_{(\delta',s)} \leq \frac{\|u\|_{(\delta,s)}}{\delta - \delta'},$$

$$\|(1-\partial_x^2)^{-1}u\|_{(\delta,s+2)} = \|u\|_{(\delta,s)} \; (p = 0,1,2),$$

$$\|(1-\partial_x^2)^{-1}\partial_x u\|_{(\delta',s+1)} \leq \frac{1}{\delta - \delta'}\|u\|_{(\delta,s)}.$$

9. time-local analytic IVP for CH system

Theorem

Let $0 < \Delta \leq 1, s \geq 2$. If $(u_0, v_0) \in \oplus^2 E_{\Delta, s+1}$, then there exists $T_\Delta > 0$ such that the IVP the CH system has a unique holomorphic solution valued in $\oplus^2 E_{\Delta d, s+1}$ in the disk $D(0, T_\Delta(1-d))$ for every $d \in]0,1[$. (t is near 0)

Method: abstract Cauchy-Kowalevsky. Scales of Banach spaces. (Ovsyannikov, Yamanaka, Trèves)

Ref: CH and similar equations, Barostichi-Himonas-Petronilho 2016

We used $\|\cdot\|_{(\delta,s)}$ to prove local analyticity in t (small).

10. New norm $\|\bullet\|_{\sigma,2}$

Next, we want to show analyticity in x (for fixed $t \in \mathbb{R}$).
Following Kato-Masuda (1986), set

$$\|f\|_{\sigma,2}^2 = \sum_{j=0}^{\infty} \frac{e^{2j\sigma}}{j!^2} \|f^{(j)}\|_2^2.$$

Do not confuse $\|\bullet\|_{\sigma,2}$ with $\|\bullet\|_{(\delta,s)}$.

$\|\bullet\|_{\sigma,2}$ is useful in the study of analytic functions:

If $f \in A(r)$, then $\|f\|_{\sigma,2} < \infty$. (Here $\sigma < \log r$)

If $\|f\|_{\sigma,2} < \infty$ for any $\sigma < \log r$, then $f \in A(r)$.

We employ $\|\bullet\|_{\sigma,2}$ to prove analyticity in x
for an arbitrarily large (fixed) t.

11. Regularity theorem by Kato and Masuda: outline

Consider the equation

$$\frac{dU}{dt} = F(U),\ U(0) = U_0.$$

Here F is typically a (nonlinear) continuous mapping from a Banach space to another.
Kato-Masuda theorem gives some sufficient condition for the regularity of $U(t), t > 0$.
If U_0 is regular to some extent, then so is $U(t), t > 0$.

Let $\{\Phi_\sigma; -\infty < \sigma < \infty\}$ be a family of functions related to norms on Banach spaces. (Liapunov family).

Φ_σ is a measure of regularity.
$\Phi_\sigma(U(t))$ can be estimated in terms of U_0.

12. Regularity theorem by Kato-Masuda: formulation

X, Z: Banach spaces and Z is a dense subspace of X.
F : continuous mapping from Z to X.
$\{\Phi_\sigma; -\infty < \sigma < \infty\}$: a family of real-valued functions on Z.
Assume

$$|\langle F(U), D\Phi_s(U)\rangle| \le K\Phi_s(U) + L\Phi_s(U)^{1/2}\partial_s\Phi_s(U) + M\partial_s\Phi_s(U).$$

D: Frechét derivative
$\langle\cdot,\cdot\rangle$ (no subscript) : the pairing of X and $\mathcal{L}(X;\mathbb{R})$.

If $dU/dt = F(U)$, $U(0) = U_0$, then for functions $s(t), r(t)$ depending on U_0 we have

$$\Phi_{s(t)}(U(t)) \le r(t),\ t \in [0,T].$$

If U_0 is regular to some extent, then so is $U(t), t > 0$.

13. Liapunov family: the case of the CH system

The system is asymmetric in (u,v)⇒asymmetric Liapunov family
Set $X = \oplus^2 H^{m+2}, \quad Z = \oplus^2 H^{m+4}$,

$$\Phi_{\sigma,m}(u,v) = \Phi^{(1)}_{\sigma,m}(u) + \Phi^{(2)}_{\sigma,m}(v),$$

$$\Phi^{(1)}_{\sigma,m}(u) = \frac{1}{2}\sum_{j=1}^{m+1} \frac{1}{j!^2} e^{2(j-1)\sigma} \frac{\|u^{(j)}\|_2^2}{2},$$

$$\Phi^{(2)}_{\sigma,m}(v) = \frac{1}{2}\sum_{j=0}^{m} \frac{1}{j!^2} e^{2j\sigma} \frac{\|v^{(j)}\|_2^2}{2}.$$

Then

$$\|u\|^2_{\sigma,2} = \|u\|_2^2 + 2\lim_{m\to\infty} e^{2\sigma}\Phi^{(1)}_{\sigma,m}(u),$$

$$\|v\|^2_{\sigma,2} = \lim_{m\to\infty} 2\Phi^{(2)}_{\sigma,m}(v)$$

and if they are finite, u and v are analytic in x.
We want to get bounds on $\Phi_{\sigma,m}(u,v)$ by using KM theory.

14. Rewriting the system

$$F(u,v) = (F_1(u,v), F_2(u,v)),$$

$$F_1(u,v) = -\beta u u_x - (1-\partial_x^2)^{-1}\partial_x \left[-\alpha u + \frac{3-\beta}{2}u^2 + \frac{\beta}{2}u_x^2 + v + \frac{1}{2}v^2\right],$$

$$F_2(u,v) = -u_x - (uv)_x.$$

Our CH system is

$$\frac{d(u,v)}{dt} = F(u,v)$$

and this is how the Kato-Masuda theory is applied.

15. Kato-Masuda and the CH system

F is a continuous mapping from $\oplus^2 H^{m+4}$ to $\oplus^2 H^{m+2}$.
There exist positive constants $K_1, K_2, L_1, L_2, M_1, M_2, M_3$ independent of u, v and σ such that we have

$$
\begin{aligned}
&|\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle| \\
&\leq [K_1 + K_2\|(u,v)\|_3]\,\Phi_{\sigma,m}(u,v) \\
&\quad + (L_1 + L_2 e^{\sigma})\,\Phi_{\sigma,m}(u,v)^{1/2}\partial_\sigma \Phi_{\sigma,m}(u,v) \\
&\quad + \left[M_1 + (M_2 + M_3 e^{2\sigma})\|(u,v)\|_3\right]\partial_\sigma \Phi_{\sigma,m}(u,v)
\end{aligned}
$$

for $(u, v) \in \oplus^2 H^{m+4}$.
Kato-Masuda theory works for $d(u,v)/dt = F(u,v)$, which is the CH system.
$\Rightarrow$ Bounds on $\Phi_{\sigma,m}(u(t), v(t))$ and regularity of the solution $(u(t), v(t))$.
$m \to \infty$ and $u(t)$ and $v(t)$ are analytic in x for any $t > 0$.

16. Estimating $\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle$

$$\langle F(u,v), D\Phi_{\sigma,m}(u,v)\rangle$$
$$= \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2}\langle u^{(j)}, \partial_x^j F_1(u,v)\rangle_2 + \sum_{j=0}^{m} \frac{e^{2j\sigma}}{j!^2}\langle v^{(j)}, \partial_x^j F_2(u,v)\rangle_2,$$

The bracket on the left-hand side is the pairing of $\oplus^2 H^{m+2}$ and its dual $(\oplus^2 H^{m+2})^* \simeq \oplus^2 H^{m+2}$.
$\langle \cdot, \cdot \rangle_2$ is the inner product of H^2.
Estimates by using

$$\|fg\|_2 \leq 8(\|f\|_2\|g\|_1 + \|f\|_1\|g\|_2) \text{ (Kato-Ponce).}$$

H^2, H^1 norms in RHS. Better than $\|fg\|_2 \leq \text{const.}\|f\|_2\|g\|_2$.

17. Estimating $\sum_{j=1}^{m+1} j!^{-2} e^{2(j-1)\sigma} \langle u^{(j)}, \partial_x^j (uu_x) \rangle_2$

$\displaystyle\sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \partial_x^j F_1 \rangle_2$ involves

$$Q_j = \sum_{\ell=1}^{j} \binom{j}{\ell} \langle u^{(j)}, u^{(\ell)} u^{(j-\ell+1)} \rangle_2. \quad \text{(degree3)}$$

Apply Schwarz and get $\|u^{(j)}\|_2 \|u^{(\ell)} u^{(j-\ell+1)}\|_2$. By Kato-Ponce,

$$\begin{aligned} \|u^{(\ell)} u^{(j-\ell+1)}\|_2 &\le 8 \left(\|u^{(\ell)}\|_2 \|u^{(j-\ell+1)}\|_1 + \|u^{(\ell)}\|_1 \|u^{(j-\ell+1)}\|_2 \right) \\ &\le 8 \left(\|u^{(\ell)}\|_2 \|u^{(j-\ell)}\|_2 + \|u^{(\ell-1)}\|_2 \|u^{(j-\ell+1)}\|_2 \right). \end{aligned}$$

We get

$$\left| \sum_{j=1}^{m+1} \frac{e^{2(j-1)\sigma}}{j!^2} \langle u^{(j)}, \partial_x^j (uu_x) \rangle_2 \right|$$

$$\le 96 \|u\|_3 \Phi_{\sigma,m}(u,v) + \left(16 \|u\|_3 + \frac{32\pi}{\sqrt{3}} e^{\sigma} \sqrt{\Phi_{\sigma,m}(u,v)} \right) \partial_\sigma \Phi_{\sigma,m}(u,v).$$

18. Final part of the proof of the main result

1. analyticity in t and x, local in t
   $\leftarrow$ Cauchy-Kowalevsky (Ovsyannikov) type argument

2. analyticity in x (arbitrarily large fixed $t > 0$)
   $\leftarrow$ Kato-Masuda, just completed

3. global analyticity in t $\leftarrow$ combination of 1 and 2